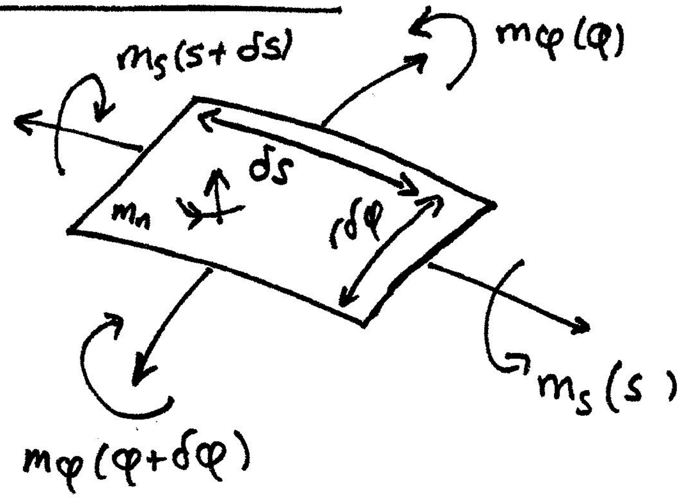
Moment balance



$$\delta s \delta\varphi \left[\frac{\partial}{\partial s}(r m_s \underline{e}_s) + \frac{\partial}{\partial \varphi}(m_\varphi \underline{e}_\varphi) + r m_n \underline{n} \right] = 0$$

In $\underline{e}_s$ direction, noting $\frac{\partial \underline{e}_\varphi}{\partial \varphi} = -\cos\theta\, \underline{e}_s - \sin\theta\, \underline{n}$

$$\boxed{\frac{\partial}{\partial s}(r m_s) - m_\varphi \cos\theta = 0}$$

$\underline{e}_\varphi$ direction
$\frac{\partial m_\varphi}{\partial \varphi} = 0$
$\underline{n}$ direction gives
m_n in terms of m_s, m_φ.

First Integral (Constant Pressure)

don't lecture

Proposition. If P is constant then $r^2(2t_s\kappa_\varphi - P) = \text{Const.}$

Proof We have $P \overset{(i)}{=} \kappa_s t_s + \kappa_\varphi t_\varphi$ $\quad \frac{\partial}{\partial s}(r t_s) \overset{(ii)}{=} t_\varphi \cos\theta$

$\frac{\partial r}{\partial s} \overset{iii}{=} \cos\theta \qquad \kappa_s \overset{iv}{=} \frac{d\theta}{ds} \qquad \kappa_\varphi \overset{(v)}{=} \frac{\sin\theta}{r}$

$$\therefore \frac{\partial}{\partial s}(r\kappa_\varphi) = \frac{\partial r}{\partial s}\kappa_\varphi + r\frac{\partial}{\partial s}\left(\frac{\sin\theta}{r}\right) = \underbrace{\frac{\partial r}{\partial s}\kappa_\varphi + r.\left(-\frac{1}{r^2}\right)\sin\theta\frac{\partial r}{\partial s}}_{0} + \frac{\partial}{\partial s}\sin\theta$$

$$= \cos\theta\frac{d\theta}{ds} = \frac{\partial r}{\partial s}\kappa_s. \quad (1)$$

$$\therefore P\frac{\partial r}{\partial s} = \frac{\partial r}{\partial s}\kappa_s t_s + \frac{\partial r}{\partial s}\kappa_\varphi t_\varphi = \frac{\partial}{\partial s}(r\kappa_\varphi)t_s + \underbrace{\frac{\partial r}{\partial s}\kappa_\varphi t_\varphi}_{\kappa_\varphi\cdot\frac{\partial r}{\partial s}t_\varphi = \kappa_\varphi\frac{\partial}{\partial s}(rt_s)}$$

(ii)
(iii)

$$\therefore Pr\frac{\partial r}{\partial s} = rt_s\frac{\partial}{\partial s}(r\kappa_\varphi) + r\kappa_\varphi\frac{\partial}{\partial s}(rt_s) = \frac{\partial}{\partial s}\left(r^2 t_s\kappa_\varphi\right)$$

$\therefore$ If P const,
$$\underline{\frac{Pr^2}{2} - r^2 t_s\kappa_\varphi = \text{Const}}$$

Constitutive Laws

We need to relate t_s and t_φ to the deformation of membrane, as summarised by the stretch ratios

$$\lambda_s = \frac{\partial s}{\partial\sigma} \qquad \lambda_\varphi = \frac{r}{\rho} \qquad \lambda_3 = \frac{1}{\lambda_s\lambda_\varphi}$$

stretch in normal direction, given by $1/\lambda_s\lambda_\varphi$
due to incompressibility

We can generally write for the tensions

$$t_s = A f_s(\lambda_s, \lambda_\varphi) \qquad t_\varphi = A f_\varphi(\lambda_s, \lambda_\varphi)$$

having eliminated $\lambda_3 = \frac{1}{\lambda_s \lambda_\varphi}$ using incompressibility.

Aside

With a hyperelastic medium, with free energy W, one can show

$$t_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - \underset{\text{pressure}}{p} \qquad i \in \{s, \varphi, 3\}$$

[e.g. Solid mechanics lecture notes eqn(233)].

With a membrane assumption, $t_3 = 0$, we have

$p = \lambda_3 \frac{\partial W}{\partial \lambda_3}$ which can be written as a function of $\lambda_s, \lambda_\varphi$ via $\lambda_3 = \frac{1}{\lambda_s \lambda_\varphi}$.

Moment constituitive relation

$$m_s = m_\varphi = M(K_s + K_\varphi - K_0)$$

Curvature of membrane in the stress free configuration, zero below

$\therefore$ Moment balance becomes

cancel as $m_s = m_\varphi$

$$\frac{\partial r}{\partial s} m_s + r \frac{\partial m_s}{\partial s} - m_\varphi \frac{\partial r}{\partial s} = 0 \quad \therefore \quad 0 = \frac{\partial m_s}{\partial s} \quad \therefore \quad 0 = \frac{\partial}{\partial s}(K_s + K_\varphi)$$

$$\therefore \quad K_s + K_\varphi = M_0, \text{const} \quad \therefore \quad K_s = M_0 - \frac{\sin\theta}{r}$$

page number change due to altered notes.

<u>In summary</u> we need z and r for the shape of the membrane. These can be generated by the following closed system of equations

$$\frac{dz}{d\sigma} = -\lambda_s \sin\theta \qquad \text{Geometry}$$

$$\frac{dr}{d\sigma} = \lambda_s \cos\theta \qquad \text{Geometry}$$

$$\frac{ds}{d\sigma} = \lambda_s \qquad \text{Geometry}$$

$$\frac{d\theta}{d\sigma} = \lambda_s K_s = \lambda_s\left(K_1 - \frac{\sin\theta}{r}\right) \qquad \text{Geometry and moment blance}$$

$$\frac{dt_s}{ds} = \lambda_s A \frac{\cos\theta}{r}\left(f_\varphi - f_s\right) \qquad \text{Force balance}$$

with $f_s = f_s(\lambda_s, r/\rho)$ $f_\varphi = f_\varphi(\lambda_s, r/\rho)$ with initial profile $z = z_0$, $r = \rho$ from which initial curvatures and thus K_1 can be found

Spheres and spherical caps are solutions for A, K_1 constant.

Evolution of the sphere or spherical cap on a slow timescale due to changes stiffness be considering the solutions genertated by changing A from its initial value, or heterogeneous A.

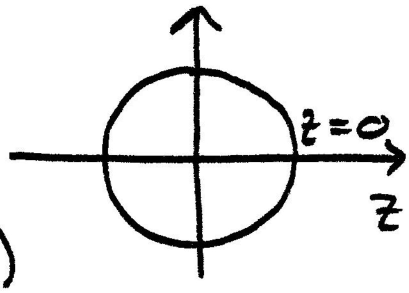
Five unknowns (z,r,s, theta, lambda_s), noting t_s = Af(lambda_s,r/rho) and five equations

Spherical Membrane

Start with a spherical membrane, radius Q and deform it to a sphere of radius q. Find the pressure in the new state given

$$t_s = Af(\lambda_s, \lambda_3) \qquad t_\varphi = Af(\lambda_\varphi, \lambda_3)$$

Old configuration geometry

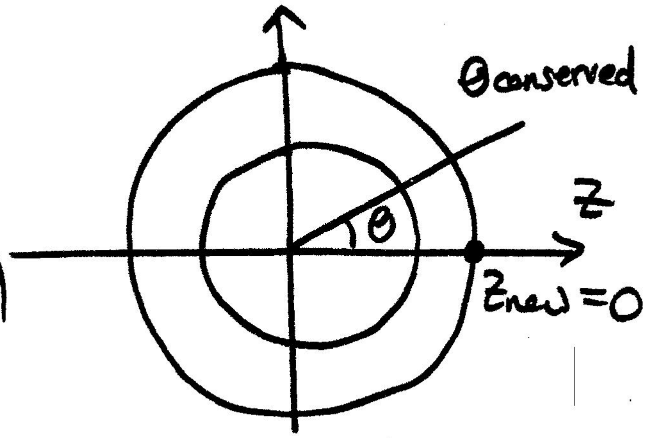


$$\sigma = Q\theta = S$$

$$z = -Q + Q\cos\theta = -Q + Q\cos(\sigma/Q)$$

$$\rho = Q\sin\theta = Q\sin(\sigma/Q)$$

New configuration geometry



$$s = q\theta$$

$$z = q(-1+\cos\theta) = q(-1+\cos(\sigma/Q))$$

$$r = q\sin(\sigma/Q)$$

$$\lambda_s = s/\sigma = q\theta/Q\theta = q/Q \, ; \quad \lambda_\varphi = \frac{r}{\rho} = \frac{q\sin(\sigma/Q)}{Q\sin(\sigma/Q)} = q/Q = \lambda_s .$$

Let $\lambda := \lambda_s = \lambda_\varphi$

$$\therefore \; t_s = t_\varphi = Af(\lambda, \lambda_3) = Af(\lambda, 1/\lambda^2)$$

$$K_s = \partial\theta/\partial s = 1/q \qquad K_\varphi = \frac{\sin\theta}{r} = 1/q$$

$$\therefore \boxed{p = K_s t_s + K_\varphi t_\varphi = 2/q \, Af(\lambda, 1/\lambda^2)} \quad \text{with } \underline{\underline{\lambda = q/Q}}.$$

Shear Stress normal to surface

If the membrane supports shear stress normal to the surface the equations become

$$\frac{dz}{d\sigma} = -\lambda_s \sin\theta \qquad \frac{d\theta}{d\sigma} = \lambda_s \kappa_s$$

$$\frac{dr}{d\sigma} = \lambda_s \cos\theta \qquad \frac{d\kappa_s}{d\sigma} = \lambda_s \left[\left[\frac{\cos\theta}{r}\left(\frac{\sin\theta}{r} - \kappa_s\right)\right] + \frac{q_s}{B} \right]$$

$$\frac{dt_s}{d\sigma} = \lambda_s A \left[\frac{\cos\theta}{r}\left(f_\varphi - f_s\right) + \kappa_s q_s / A \right]$$

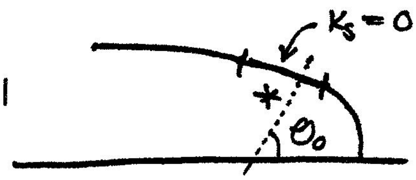
new terms.

$$\frac{dq_s}{d\sigma} = \lambda_s A \left[P/A - \kappa_s f_s - \frac{\sin\theta}{r} f_\varphi - \frac{q_s \cos\theta}{Ar} \right]$$

Pressure difference across membrane.

Exercise Show that an undeformed axisymmetric cell with a homogeneous membrane and a constant pressure across its membrane cannot have a patch of membrane with $\kappa_s = 0$

unless it's perpendicular or parallel to the axis of symmetry



Solution Undeformed $\therefore s = \sigma$. $\therefore \lambda_s = 1$

In region $*$ $\quad z = z_0 - \sin\theta_0\,\sigma$ $\qquad \kappa_s = \dfrac{d\theta_0}{ds} \equiv 0.$

$$r = r_0 + \cos\theta_0\,\sigma$$

$$\therefore q_s = \frac{-B\cos\theta_0\sin\theta_0}{(r_0+\sigma\cos\theta_0)^2}$$

($\therefore$ Contradiction at this stage for Model without normal shear stress)

$$\therefore \frac{dq_s}{d\sigma} = \frac{+2B\cos^2\theta_0\sin\theta_0}{(r_0+\sigma\cos\theta_0)^3} = \left[P - \frac{\sin\theta_0 \; A}{(r_0+\cos\theta_0\sigma)} \underbrace{f_\varphi(1,1)}_{\text{Undeformed}} + \frac{B\cos^2\theta_0\sin\theta_0}{(r_0+\sigma\cos\theta_0)^3}\right]$$

$$\therefore P = \frac{\sin\theta_0\, f_\varphi(1,1)}{(r_0+\cos\theta_0\,\sigma)} + \frac{B\cos^2\theta_0\sin\theta_0}{(r_0+\sigma\cos\theta_0)^3}$$

As required, ※. LHS constant, RHS never constant.

III Examples of 3D Biomechanics from biofluids

1. Newtonian fluid mechanics + Navier Stokes

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u}$$

$$\nabla\cdot u = 0$$

Assumes
$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Non-dimensionalise $t = Tt'$ $\quad u = \mathcal{U}u'$ $\quad \underline{x} = L\underline{x}'$ $\quad p = \mathcal{U}\mu/L\, p'$

$$\beta\frac{\partial \underline{u}'}{\partial t'} + Re\, \underline{u}'\cdot\nabla'\underline{u}' = -\nabla' p' + \nabla'^2\underline{u}'$$

$\beta = \frac{\rho L^2}{\mu T}$ $\qquad Re = \frac{\rho \mathcal{U} L}{\mu}$

For $L/T \sim \mathcal{U}$, which is true if the fluid is being forced by a boundary $\beta \sim Re$.

Exceptions :

Bead
Oscillate
fast

$u = \mathcal{U}\sin(\omega t/\varepsilon)$, then $\beta \gg Re$

For a beating cilium or flagellum

$L \sim 10\mu m$ $\quad \mathcal{U} \sim 100\mu m\, s^{-1}$ $\quad T \sim 0.1 sec$ $\quad \rho/\mu \sim 10^6$ SI units.

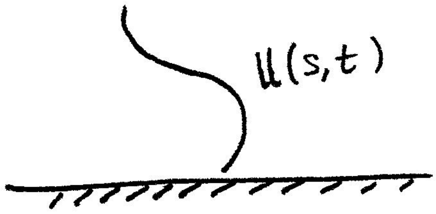
$$\beta \sim Re \sim 10^6 . 10^{-4} . 10^{-5} \simeq \underline{\underline{10^{-3}}}$$

∴ We only consider

Stokes' Equations (Dropping Primes)

$$0 = -\nabla p + \nabla^2 \underline{u}$$
$$0 = \nabla \cdot \underline{u}$$

Cilium



$0 = -\nabla p + \nabla^2 \underline{u}$ u=0 on substrate

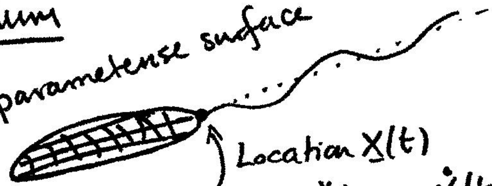
$0 = \nabla \cdot \underline{u}$

$\underline{u} = \underline{U}(s,t)$ on cilium surface, with $\underline{U}$ known.

[infinitesimal thickness]

Flagellum

$\underline{x}(t,p,q)$ parameterise surface



Location $\underline{X}(t)$

Velocity $\underline{U}^*(t), \equiv \dot{\underline{X}}(t)$

Ang Velocity $\underline{\Omega}(t)$

a priori unknown.

$$\left.\begin{aligned} 0 &= -\nabla p + \nabla^2 \underline{u} \\ 0 &= \nabla \cdot \underline{u} \end{aligned}\right\} \text{external to cell.}$$

$$\underline{u} = \underline{U}^*(t) + (\underline{x}(t,p,q) - \underline{X}(t)) \wedge \underline{\Omega}(t) + \begin{cases} \underline{U}^{flag}\overset{\swarrow \text{known}}{(t,p,q)} & \text{On flagellum} \\ \underline{0} & \text{On head} \end{cases}$$

Need 6 extra constraints as we need to find $\underline{U}^*(t)$, $\underline{\Omega}(t)$.

Swimmer neutrally buoyant

Dimensional Force Balance

$$\underbrace{M \ddot{\underline{X}}_g(t)}_{\text{Acceleration of centre of mass}} = \int_{\text{Surface of swimmer}} \sigma_{ij} n_j \, dS$$

Ratio of terms

$$\frac{\frac{ML}{T^2}}{L^2 \left[\frac{\mu U}{L}\right]} \quad \leftarrow \text{Scale of } \underline{\underline{\sigma}}$$

$$= \frac{\rho L^4/T^2}{\mu U L} = \frac{\rho U^2 L^2}{\mu U L}$$

$$= \frac{\rho U L}{\mu} = Re \ll 1$$

$\therefore$ <u>In small Re limit</u>

$$\int_{\substack{\text{Surface}\\ \text{of Swimmer}}} \sigma_{ij} n_j \, dS = 0, \quad \text{no net force on swimmer}$$

(3 extra constraints)

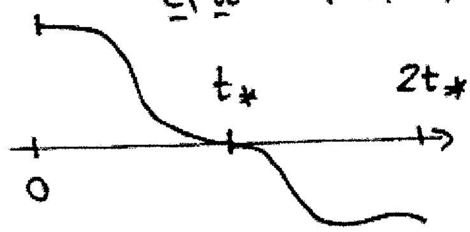
<u>Similarly</u>, No net torque on swimmer (another 3 constraints)

<u>These 6 constraints close the system.</u>

<u>Purcell's theorem</u>

Consider a swimmer in the zero Reynolds number limit. (or flagellar beat pattern)
A time reversible swimming stroke
generates no net motion.

<u>Proof</u>



time reversible means $\quad \underline{u}^{flag}(2t_* - t, p, q) = -\underline{u}^{flag}(t, p, q)$

By linearity
$$\underline{u}(\underline{x}, t) = -\underline{u}(\underline{x}, 2t^* - t)$$
$$\underline{U}^*(t) = -\underline{U}(2t^* - t)$$
$$\underline{\Omega}(t) = -\underline{\Omega}(2t^* - t)$$
$$\underline{X}(t) = \underline{X}(2t^* - t) \quad \left\{ \begin{array}{l}\text{Integral of odd}\\ \text{function is even}\end{array} \right\}.$$

a solution. By uniqueness (not proved, but standard) it is the solution $\therefore$ $\underline{X}(0) = \underline{X}(2t^*)$ $\therefore$ No net motion.