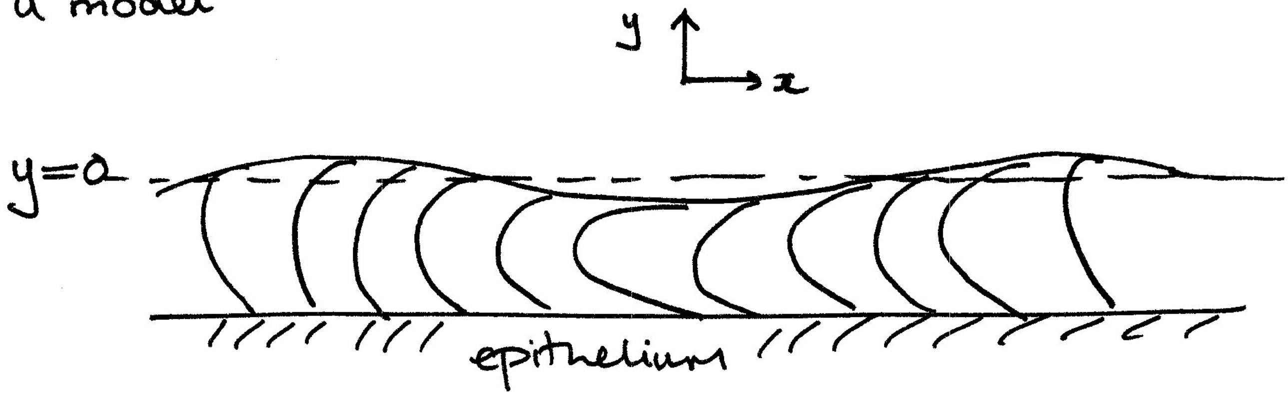
Cilary Pumping

Ciliary beating in the lung drives fluid above the cilia up the airway, resulting in mucociliary pumping. As a model



we consider the envelope formed by the cilia, which is given by

$$x_e = x + \varepsilon a \cos(x-t)$$
$$y_e = \varepsilon b \sin(x-t)$$

where $y_e = 0$ is the midplane of the envelope. Hence on the envelope

$$u_e = \varepsilon a \sin(x-t)$$
$$v_e = -\varepsilon b \cos(x-t)$$

with $(u,v) \to (U,0)$ as $y \to \infty$, with U unknown a-priori.

∴ We have $\qquad \nabla \cdot \underline{u} = 0 \qquad 0 = -\nabla p + \nabla^2 \underline{u}$

with $(u,v) = (u_e, v_e)$ on $(x,y) = (x_e, y_e)$

$(u,v) \to (U,0)$ as $y \to \infty$; U to be found.

$\nabla \cdot u = 0$ and 2D $\therefore$ $u = \psi_y \quad v = -\psi_x$ where ψ is a streamfunction.

$$0 = \nabla \wedge \nabla^2 \underline{u} = \nabla^2 (\nabla \wedge \underline{u})$$

$$\nabla \wedge \underline{u} = \begin{vmatrix} \underline{e_1} & \underline{e_2} & \underline{e_3} \\ \partial_x & \partial_y & \partial_z \\ \psi_y(x,y) & -\psi_x(x,y) & 0 \end{vmatrix} = -\nabla^2 \psi \, \underline{e_3}$$

$\therefore \; \underline{\underline{\nabla^4 \psi = 0}}$

<u>Seek a perturbative solution</u>

If $\varepsilon = 0$, nothing driving flow $\therefore$ $\underline{u} = 0$, $\left\{ \begin{matrix} u = 0 \\ \psi = 0 \end{matrix} \right\}$. Hence we try

$$\psi = \varepsilon \psi_1(x,y) + \varepsilon^2 \psi_2(x,y) + O(\varepsilon^3)$$
$$U = \varepsilon U_1 + \varepsilon^2 U_2 + O(\varepsilon^3)$$

$$\therefore \quad 0 = \nabla^4 \psi = (\partial_{xx} + \partial_{yy})^2 \psi$$

<u>$O(\varepsilon^1)$</u> $\qquad 0 = \nabla^4 \psi_1$

On $\left. \begin{matrix} x = x_e \\ y = y_e \end{matrix} \right\}$ $\quad \psi_y(x_e, y_e) = u_e = \varepsilon a \sin(x-t)$

$\qquad \psi_x(x_e, y_e) = -v_e = \varepsilon b \cos(x-t)$

$$\therefore \quad \psi_{1y}(x,0) + h.o.t = a \sin(x-t)$$
$$\psi_{1x}(x,0) + h.o.t = b \cos(x-t)$$

$$\lim_{y \to \infty} \psi_{1y}(x,y) = U_1 \qquad \lim_{y \to \infty} \psi_{1x}(x,y) = 0$$

$\therefore$ Let $\psi_1 = f(y)\sin(x-t)$

$$0 = \nabla^4\psi_1 = (\partial_{xx}+\partial_{yy})\left[f''-f\right]\sin(x-t) = \left[f''''-f''\right]\sin(x-t) - \left[f''-f\right]\sin(x-t)$$

$$= \left[f''''-2f''+f\right]\sin(x-t) \quad \therefore \; f \propto e^{my}$$

$$(m^2)^2 - 2m^2 + 1 = 0$$
$$(m^2-1)^2 = 0$$
$$\therefore m^2 = 1 \;\therefore\; \underline{\underline{m = \pm 1}}$$

$$\therefore f = Ae^{y} + Be^{-y} + Cye^{y} + Dye^{-y}$$

Need $\partial f/\partial y$ finite as $y \to \infty$ $\quad \therefore \; f = (B+Cy)e^{-y}$

$$\left.\begin{aligned} \therefore \quad \psi_1 &= (B+Cy)e^{-y}\sin(x-t) \\ \therefore \quad \psi_{1y} &= (-B-Cy+C)e^{-y}\sin(x-t) \\ \psi_{1x} &= (B+Cy)e^{-y}\cos(x-t) \end{aligned}\right\} \therefore \; \underline{\text{on } y=0} \quad \underline{\underline{B = b}}$$

$$-B + C = -b + C = a$$
$$\therefore C = a + b.$$

$$\therefore \quad \boxed{\psi_1 = (b + (a+b)y)e^{-y}\sin(x-t)} \quad \therefore \; \underline{U_1 = 0 \text{ by matching at } \infty}$$

$\therefore$ We need to consider higher orders

$\underline{O(\varepsilon^2)}$

$$a\cos(x-t)\psi_{1yx}(x,0) + b\sin(x-t)\psi_{1yy} + \psi_{2y}(x,0) = 0$$

$$a\cos(x-t)\psi_{1xx}(x,0) + b\sin(x-t)\psi_{1yx} + \psi_{2x}(x,0) = 0$$

$$\therefore \psi_{2y}(x,0) = -a^2\cos^2(x-t) + b(b+2a)\sin^2(x-t)$$

$$\psi_{2x}(x,0) = +ab\sin(x-t)\cos(x-t) - ab\cos(x-t)\sin(x-t) = 0$$

$$\therefore \quad \psi_{2y} = \tfrac{1}{2}\left(b^2 + 2ab - a^2\right) + \underbrace{\gamma\cos\left(2(x-t)\right)}$$

$$\psi_{2x} = 0 \qquad \text{on } y = 0$$

Linearity... this will generate terms that will not contribute to U.

$\therefore$ Consider only η where $\quad \nabla^4\eta = 0$

$\eta_y = \frac{1}{2}(b^2 + 2ab - a^2)$ on $(x,0)$

$\eta_x = 0$ on $(x,0)$

$\eta_y \to U$ to be determined as $y \to \infty$.

Let $\eta = g(y)$. $\qquad \eta_y = \text{Const}$ a solution $\quad \therefore \eta_y = \frac{1}{2}(b^2 + 2ab - a^2) \equiv U$ is a solution.

Changing η by a constant has no effect; higher powers of y don't match BC

$$\therefore \boxed{U_2 = \tfrac{1}{2}\left(b^2 + 2ab - a^2\right)} \quad \therefore U = \tfrac{1}{2}\left(b^2 + 2ab - a^2\right)\varepsilon^2 + \text{h.o.t.}$$

Cellular Motility

Singularity solutions for Stokes Equations

The point force solution... the Stokeslet

Stokes Equations are linear... we can build up solutions in terms of point force solutions, dipoles, quadrapoles etc, as in electrostatics for example

Suppose a point force of strength $\underline{m}$ is located at $\underline{x}_0$. Then

$$-\nabla p + \mu \nabla^2 \underline{u} + \underline{m}\,\delta(\underline{x}-\underline{x}_0) = 0 \qquad \nabla\cdot\underline{u} = 0$$

By translational invariance, WLOG $\underline{x}_0 = \underline{0}$. Let $r = |\underline{x}|$.

$$\delta(\underline{x}) = -\tfrac{1}{4\pi}\nabla^2(1/r) \quad \text{as} \quad \int_{r<\varepsilon} \nabla^2(1/r)\,dV = \int_{r=\varepsilon} \underline{n}\cdot\nabla(1/r)\,dS = -4\pi.$$

$$\therefore \quad -\nabla^2 p + m\cdot\nabla\delta(\underline{x}) = 0 = -\nabla^2 p + \underline{m}\cdot\nabla\left(-\tfrac{1}{4\pi}\nabla^2\left(\tfrac{1}{r}\right)\right)$$

$$\therefore \quad p = -\tfrac{1}{4\pi}\,\underline{m}\cdot\nabla(1/r)$$

$$\therefore \quad \mu\nabla^2\underline{u} = \underline{m}/4\pi\,\nabla^2(1/r) - \tfrac{1}{4\pi}\nabla(\underline{m}\cdot\nabla(1/r))$$

$$\therefore \quad \mu\nabla^2 u_i = \frac{m_j}{4\pi}\left[\delta_{ij}\nabla^2 - \frac{\partial^2}{\partial x_i\partial x_j}\right]\frac{1}{r}$$

Suppose $\nabla^2 G(r) = +1/r$.

Then $\nabla^2 \left\{ \frac{m_j}{4\pi} \left[\delta_{ij} \nabla^2 - \frac{\partial^2}{\partial x_i \partial x_j} \right] G(r) \right\} = \frac{-m_j}{4\pi} \left(\delta_{ij} \nabla^2 - \frac{\partial^2}{\partial x_i \partial x_j} \right) \left(\frac{1}{r} \right)$

$\therefore$ Let $u_i = + \frac{1}{\mu} \frac{m_j}{4\pi} \left(\delta_{ij} \nabla^2 - \frac{\partial^2}{\partial x_i \partial x_j} \right) G.$

Solve $\quad \frac{1}{r}(rG)'' = -\frac{1}{r} \quad \therefore (rG)'' = +1 \quad \therefore (rG) = +r^2/2 + Ar + B$

$\therefore \quad G = +r/2 + A + B/r$

Wiped out by derivatives.

$\therefore G = -r/2$.

$\therefore \quad u_i = \frac{1}{8\pi\mu} m_j \left(\delta_{ij} \nabla^2 - \frac{\partial^2}{\partial x_i \partial x_j} \right) r$

$= \frac{m_j}{8\pi\mu} \left(\delta_{ij}/r + \frac{r_i r_j}{r^3} \right) = \frac{G\_ijm\_j}{8 \text{ Pi Mu}}$

Dims of r^2... no constant to make this up. (either r^2 or $\frac{p}{|m|} \propto 1/r^2$).

To reinstate x_0 dependence use $r_i = (x - x_0)_i$ and $r = |x - x_0|$

Differentiating this solution, and other singularity solutions, wrt x_0 generates many more solutions.

Derivatives wrt x_0 also generate solutions to Stokes' equations, with singular forcings.

Superposition of such solutions provides an easy means of solving Stokes flow problems.

Potential Dipole Solution

Firstly
$$p = \text{const}$$
$$u_i = -\frac{\partial}{\partial x_i} \frac{1}{|\underline{x} - \underline{x}_0|} = \frac{(\underline{x} - \underline{x}_0)}{r^3}$$

$$r = |\underline{x} - \underline{x}_0|$$

is a solution of Stokes' Equation. Hence $\nabla^2 u = 0$, $\nabla \cdot u = 0$

Hence with
$$D_{ij} = \frac{\partial}{(\partial x_0)_j}\left(\frac{x_i - (x_0)_i}{|\underline{x} - \underline{x}|^3}\right) = -\frac{\delta_{ij}}{r^3} + \frac{3\hat{x}_i\hat{x}_j}{r^5}$$
where
$r = |\underline{x} - \underline{x}_0|$ $\quad \hat{\underline{x}} = \underline{x} - \underline{x}_0$.

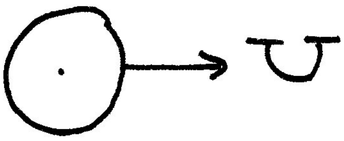
Thus $\quad u_i = D_{ij} q_j \quad$ for $\underline{q}$ a constant vector is a solution of Stokes equation

$$W_i = \frac{\partial u_i}{\partial x_{0j}} q_j$$

$$\frac{\partial w_i}{\partial x_i} = \frac{\partial}{\partial x_{0j}}\left(\frac{\partial u_i}{\partial x_i}\right) q_j = 0 \qquad \text{as } \nabla \cdot u = 0$$

$$\nabla^2 W_i = \nabla^2\left(\frac{\partial u_i}{\partial x_{0j}}\right) q_j = \frac{\partial(\nabla^2 u_i)}{\partial x_{0j}} q_j = 0 \qquad \text{as } \nabla^2 \underline{u} = 0$$

Solution for translating sphere



At instantaneous time point, sphere centre at ~~origin~~ $\underline{x}_0$

$$\left.\begin{aligned} -\nabla p + \mu \nabla^2 u &= 0 \\ \nabla \cdot u &= 0 \end{aligned}\right\} \quad |\underline{x} - x_0| > a$$

$$\underline{u} = \underline{U} = U e_{\underline{x}}$$

on $|\underline{x} - \underline{x}_0| = a$.

Consider $\quad u_i = u_i^{(1)} + u_i^{(2)}$

$u_i^{(1)} = G_{ij} g_j \qquad u_i^{(2)} = D_{ij} q_j$

$\therefore$ On $r = a$

$$U_i = u_i = \left(\delta_{ij}/a + \frac{\hat{x}_i \hat{x}_j}{a^3}\right) g_j + \left(-\delta_{ij}/a^3 + \frac{3\hat{x}_i \hat{x}_j}{a^5}\right) q_j$$

$$= \left(g_i/a - q_i/a^3\right) + \frac{\hat{x}_i \hat{x}_j}{a^3}\left(g_j + \frac{3}{a^2} q_j\right)$$

$\therefore$ Let $\underline{q} = -\frac{a^2}{3}\underline{g}$

$$\therefore \quad \underline{U} = \frac{1}{a}\left(\underline{g} - \frac{1}{a^2}\left(-\frac{a^2}{3}\underline{g}\right)\right) = \frac{4}{3a}\underline{g} \quad \therefore \underline{g} = \frac{3a}{4}\underline{U}$$

$$\therefore \boxed{\underline{g} = \frac{1}{8\pi\mu}\left(6\pi\mu a \underline{U}\right) = -3/a^2 \, \underline{q}}$$

$\therefore$ We have the solution for the translating sphere

Drag Force Potential Dipole does not contribute to the force

$\therefore D_{ij} q_j \sim O(1/r^3)$ O

$$\int_{\text{Sphere}} \sigma_{ij} n_j \, dS = \int_{S_\infty} \sigma_{ij} n_j \, dS \sim O\left(1/r^4 \cdot r^2\right) \sim O(1/r^2) \rightarrow 0.$$