1 Review of core complex analysis

1.1 Introduction

Here we summarise the important results and techniques of core complex analysis that are assumed for the remainder of this course. This is by no means a complete account of complex analysis. Many important details and most of the proofs will be omitted, but may be found in basic textbooks (e.g. Priestley). First we introduce some basic notation that will be used throughout this course.

Complex conjugation is denoted by an overbar: if x, $y \in \mathbb{R}$ and $z = x + \mathrm{i}y$, then $\overline{z} = x - \mathrm{i}y$. Conjugation commutes with all of the basic complex functions, that is, $\overline{z^k} \equiv \overline{z}^k$, $\overline{\mathrm{e}^z} \equiv \mathrm{e}^{\overline{z}}$, $\overline{\sin z} \equiv \sin \overline{z}$, etc.

A **region** in the complex plane, usually denoted D, is an open, path-connected subset of $\mathbb{C}$, simply-connected unless stated otherwise. Its boundary is denoted ∂D.

A **contour** Γ is a simple, piecewise continuously differentiable path in $\mathbb{C}$ with the positive (anti-clockwise) orientation (a *Jordan contour*), closed unless stated otherwise.

A **disc** in the complex plane is denoted by $D(a; R) = \{z \in \mathbb{C} : |z - a| < R\}$, i.e. the open disc centre a and radius R.

1.2 Holomorphic functions

A function $f(z)$ of the complex variable z is said to be *differentiable* at the point z if $\lim_{h\to 0}\big(f(z+h) - f(z)\big)/h$ exists, independent of how $h \to 0$. When this is true, we define the derivative

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}. \tag{1.1}$$

If $f(z)$ is differentiable at each point in a region D, then f is said to be *holomorphic* in D.

Suppose we decompose a holomorphic function into its real and imaginary parts by writing

$$f(z) = u(x, y) + \mathrm{i}v(x, y), \tag{1.2}$$

where $z = x + \mathrm{i}y$. Then, by taking h first real then imaginary in (1.1) and setting the two resulting values of $f'(z)$ equal, we find that u and v must satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.3}$$

The real and imaginary parts of any holomorphic function must satisfy (1.3). The reverse is not exactly true. If u and v are differentiable as functions of (x, y) *and* satisfy the Cauchy–Riemann equations (1.3), then $f = u(x, y) + \mathrm{i}v(x, y)$ is a holomorphic function of $z = x + \mathrm{i}y$.

By cross-differentiating the Cauchy–Riemann equations (1.3), we quickly find that both u and v satisfy Laplace's equation, i.e.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \nabla^2 v = 0. \tag{1.4}$$

The reverse is also true: if a function $u(x, y)$ satisfies Laplace's equation, then it is the real part of some function $f(z)$ that is holomorphic (at least locally). Therefore solving Laplace's equation in two dimensions is equivalent to finding a suitable holomorphic function of z. Many of the applications covered in this course spring from this basic fact.

If one were to think of a general (not necessarily holomorphic) complex-valued function $G(x, y)$, it could equivalently be viewed as a function $g(z, \overline{z})$ using the one-to-one correspondence between $(z, \overline{z})$ and (x, y). The chain rules relating partial derivatives with respect to the two sets of coordinates are given by

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \qquad \frac{\partial}{\partial y} = \mathrm{i}\frac{\partial}{\partial z} - \mathrm{i}\frac{\partial}{\partial \overline{z}}, \tag{1.5}$$

which can easily be inverted to

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right). \tag{1.6}$$

From these and the Cauchy–Riemann equations (1.3), it follows (Exercise 1 of sheet 1) that if g is holomorphic, then

$$\frac{\partial g}{\partial \overline{z}} = 0. \tag{1.7}$$

This result may be interpreted as saying that a holomorphic function is one that is independent of $\overline{z}$.

Given a holomorphic function $f(z)$, we define a new function $\bar{f}(z)$ as follows:

$$\overline{f}(z) := \overline{f(\overline{z})} \tag{1.8}$$

(note the double conjugation). The Cauchy–Riemann equations can be used to show that, if $f(z)$ is holomorphic, then so is $\overline{f}(z)$. The mapping from $f(z)$ to $\overline{f}(z)$ is called *Schwarz reflection.* It is easily seen that a repetition of Schwarz reflection returns the original function f, i.e. $\overline{\overline{f}}(z) \equiv f(z)$.

1.3 Path integrals and Cauchy's Theorem

The integral of a complex valued function of z along a curve Γ (which may be open or closed) in $\mathbb{C}$, is defined as usual: if Γ is parameterised using a real-valued parameter t, i.e. $\Gamma = \{z(t) : t \in (t_0, t_1)\}$, then

$$\int_\Gamma g(z, \overline{z})\,\mathrm{d}z := \int_{t_0}^{t_1} g\big(z(t), \overline{z(t)}\big)\, z'(t)\,\mathrm{d}t. \tag{1.9}$$

Such an integral can easily be bounded using

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \le \mathrm{length}(\Gamma) \times \sup_{z\in\Gamma} |f(z)|\,. \tag{1.10}$$

Cauchy's Theorem: *If a function $f(z)$ is holomorphic within a simple closed contour Γ, and continuous on Γ, then*

$$\int_\Gamma f(z)\,\mathrm{d}z = 0. \tag{1.11}$$

At first glance, Cauchy's theorem might appear very mysterious. However, if we write $f = u + \mathrm{i}v$ and then use Green's Theorem, then Cauchy's Theorem follows directly from the Cauchy–Riemann equations (1.3). However, this "proof" assumes that u and v have continuous partial derivatives on and inside Γ. A much more technical proof, due to Goursat, assumes only that $f'(z)$ exists everywhere inside Γ. This is significant because, as we shall see, one can then *prove* that $f'(z)$ is continuous, rather than assuming so.

It is an immediate consequence of Cauchy's theorem that if Γ_1 and Γ_2 are two curves joining the point z_0 to another point z_1, as illustrated in Figure 1.1, and if $f(z)$ is holomorphic in a region containing Γ_1, Γ_2 and the region between them, then

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z, \tag{1.12}$$

so that the integral is path-independent. This is often stated as the **deformation theorem**: if one contour Γ_1 can be deformed smoothly into another one Γ_2 while crossing only points at which $f(z)$ is holomorphic, then the integral of $f(z)$ along Γ_1 is equal to the integral along Γ_2. It also allows us to define an anti-derivative of $f(z)$ by the prescription

$$F(z) = \int_{z_0}^{z} f(t)\,\mathrm{d}t, \tag{1.13}$$

provided that we do so in a simply connected region within which $f(z)$ is holomorphic, as the contour of integration is immaterial.

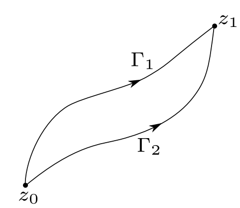

Figure 1.1: Two paths from z_0 to z_1.

A partial inverse of Cauchy's Theorem is:

Morera's Theorem. *If $f(z)$ is continuous in D and $\oint_\Gamma f(z)\mathrm{d}z = 0$ for all simple closed Γ in D, then $f(z)$ is holomorphic in D.*

1.4 Cauchy's integral formula

Take a simple closed contour Γ, and let $f(z)$ be holomorphic on Γ and inside it. Then values of $f(z)$ on Γ determine its values at all points within Γ as well, via **Cauchy's integral formula**: for all z within Γ,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t. \tag{1.14}$$

The proof is simple, by deforming the contour to a small circle surrounding z, as shown in Figure 1.2, and adding and subtracting $f(z)$:

$$\oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t = \oint_{|t-z|=\epsilon} \frac{f(t)}{t-z}\,\mathrm{d}t = \oint_{|t-z|=\epsilon} \frac{f(z)}{t-z}\,\mathrm{d}t + \oint_{|t-z|=\epsilon} \frac{f(t)-f(z)}{t-z}\,\mathrm{d}t; \tag{1.15}$$

the first integral on the right is equal to $2\pi\mathrm{i}f(z)$ and the second vanishes as $\epsilon \to 0$ by continuity of f.

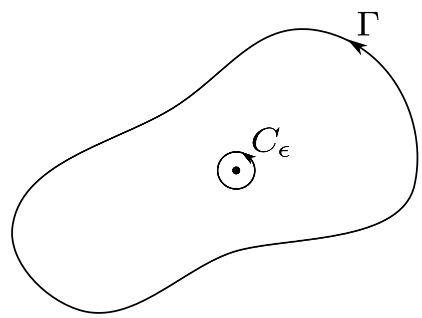

Figure 1.2: Deformation of Γ to $C_\epsilon = \{z : |t-z| = \epsilon\}$.

With $f(z)$ given by Cauchy's integral formula (1.14), it is tempting to differentiate with respect to z under the integral sign to find

$$f'(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^2}\,\mathrm{d}t, \tag{1.16}$$

and indeed the formal justification of this step is not difficult. Thus the value of $f'(z)$ at any point inside Γ may be expressed solely in terms of the values taken by $f(z)$ on Γ. But then, we can differentiate again (with essentially the same justification), to find an analogous formula for $f''(z)$, namely

$$f''(z) = \frac{1}{\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^3}\,\mathrm{d}t. \tag{1.17}$$

Recall we have assumed only that $f(z)$ is holomorphic. Therefore we have established that, given only that $f'(z)$ exists (not even that it is continuous), it follows that $f''(z)$ also exists. By iterating on this process, we obtain **Cauchy's formula for derivatives:**

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^{n+1}}\,\mathrm{d}t. \tag{1.18}$$

Hence, a holomorphic function is infinitely differentiable, in marked contrast with real-valued functions.

A function is called *entire* if it is holomorphic in the whole complex plane (e.g. z, e^z). Such a function must have a singularity at infinity, because of:

Liouville's Theorem: *Any bounded entire function $f(z)$ is constant.*

That is, if $|f(z)| < M$ for some M and for all z, then f is a constant. Liouville's Theorem may be proved by looking at Cauchy's integral formula (1.16) for $f'(z)$ and taking Γ to be a large circle; letting the radius of the circle tend to infinity, we deduce that $f'(z) = 0$.

There are various generalisations of Liouville's Theorem which apply when the modulus of an entire function is bounded in some specified way (not just by a constant) as $z \to \infty$. The simplest generalisation is

Corollary: *If $f(z)$ is entire and $f(z) = O(z^n)$ as $z \to \infty$ for $n \in \mathbb{N}$, then $f(z)$ is a polynomial of degree n.*

This result is easily proved by applying Liouville's Theorem to $f^{(n)}(z)$.

1.5 Taylor's Theorem and analytic continuation

Knowing that a holomorphic function has derivatives of all orders, we expect it to have power series representation.

Taylor's theorem. *If $f(z)$ is holomorphic in a disc $D(a; R)$, then there is a series representation*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}, \tag{1.19}$$

and the series converges to $f(z)$ for all $0 \le |z-a| < R$.

The proof of Taylor's Theorem involves expanding the integrand in Cauchy's integral formula (1.14) using the Binomial Theorem and then integrating term by term (which is justified by uniform convergence).

Taylor's Theorem tells us that a holomorphic function is *analytic*: it can be represented by a convergent Taylor series. The *radius of convergence* of the series (1.19) is the largest possible value of R such that the series converges, i.e. the largest value of R such that $f(z)$ is holomorphic in $D(a; R)$. In other words, the radius of convergence is the distance from $z = a$ to the nearest singular point of $f(z)$. The power series (1.19) diverges for $|z - a| > R$, while on the *circle of convergence* $|z - a| = R$ it may converge at some points, but must diverge at at least one.

A simple example is the function $1/(1 - z)$, which is holomorphic except at the point $z = 1$. Therefore it is holomorphic on the disc $D(0; 1)$ and has a Taylor expansion about $z = 0$, namely

$$f(z) = \sum_{n=0}^{\infty} z^n, \tag{1.20}$$

which converges for $|z| < 1$. But if we *define* a function $f(z)$ by the power series (1.20), then the series *diverges*, so that $f(z)$ does not even exist, for $|z| > 1$. The function $f(z)$ defined by the power series (1.20) can be *analytically continued* onto a set outside its disc of convergence by defining a new function as follows:

$$\tilde{f}(z) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} z^n & |z| < 1, \\[2ex] \dfrac{1}{1-z} & |z| \ge 1,\ z \ne 1. \end{cases} \tag{1.21}$$

The resulting function $\tilde{f}(z)$ is holomorphic on the extended set $z \in \mathbb{C} \setminus \{1\}$. The **Identity Theorem** below tells us that this continuation of $f(z)$ outside its disc of convergence is unique.

In the above example (1.21), we knew the precise location of the singularity in $\tilde{f}(z)$ and indeed had an exact formula for the analytic continution into $|z| > 1$. More generally, to get outside the circle of convergence one has to form another series centred at a point near the boundary of the original circle, and hope that the new series converges in a disc containing points outside the original circle of convergence. The process is then repeated with a new circle, and so on, as illustrated in Figure 1.3. By repeating this process we hope to analytically continue the original function into an ever larger region of the complex plane.

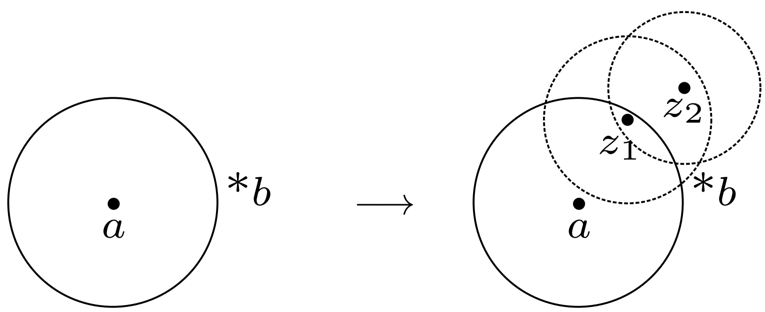

Figure 1.3: Analytic continuation of $f(z)$ out of its original circle of convergence. The radius of convergence is the distance from $z = a$ to $z = b$, the nearest singular point of $f(z)$.

However, it should not be thought that analytic continuation is automatically possible. For example, the function

$$f(z) = \sum_{n=0}^{\infty} z^{n!} \tag{1.22}$$

is holomorphic for $|z| < 1$, by comparison with the geometric series, but the sum diverges at all points $z = \mathrm{e}^{\mathrm{i}\theta}$ for which θ is a rational multiple of 2π. Thus the series has a dense set of singularities on the unit circle, and the unit circle is said to be a *natural boundary*, across which it is impossible to analytically continue the function (1.22).

The following theorem guarantees that, if an analytic continuation of a function can be found, then it is unique, locally at least.

Identity theorem. *Suppose $f_1(z)$ and $f_2(z)$ are both holomorphic in D. If there is a sequence of points $z_n \in D$, having an accumulation point which also lies in D, such that $f_1(z_n) = f_2(z_n)$, then $f_1(z) \equiv f_2(z)$ in D.*

An alternative version of the theorem is that if f_1 and f_2 agree on a dense set, then they agree everywhere. Note that it is important that the accumulation point is also in D.

A consequence of the identity theorem is that the zeroes of a non-constant function $f(z)$ holomorphic in D are isolated, in that they cannot have an accumulation point in D; if they did, then $f(z)$ would be zero by the identity theorem.

The Identity Theorem implies that functions generated by analytic continuation are locally unique. Globally, if we analytically continue a function out of an initial circle of convergence using two different chains of circles, to arrive at the same exterior point by two different routes, the continued value is the same provided that the function is holomorphic in the region between the two chains, a result called the *Monodromy Theorem*, illustrated in Figure 1.4. If not, we may end up generating branches of a multi-function such as $\log z$.

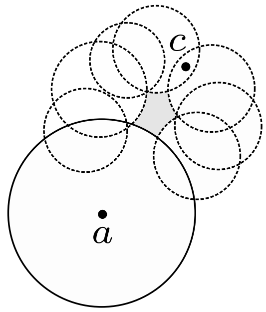

Figure 1.4: *Monodromy Theorem*: $f(z)$ is single-valued at $z = c$ provided $f(z)$ is holomorphic in the shaded area.

1.6 Laurent's Theorem and isolated singularities

Laurent's theorem. *If $f(z)$ is holomorphic in the annulus $S < |z-a| < R$, then in that annulus it has a series representation*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad \textit{where} \quad c_n = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-a)^{n+1}}\,\mathrm{d}t. \tag{1.23}$$

The integration contour Γ is any simple closed path that encloses $z = a$ and lies entirely in the annulus $S < |z-a| < R$.

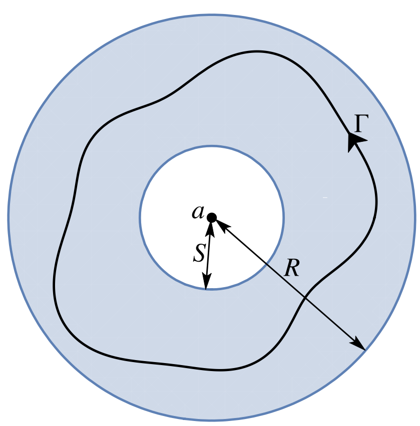

Figure 1.5: The integration contour Γ in Laurent's Theorem.

The integration contour Γ is illustrated in Figure 1.5. The part of the sum (1.23) containing negative powers of z, i.e. $\sum_{n=-\infty}^{-1} c_n (z-a)^n$, is called the *principal part* of $f(z)$ at $z = a$, and it is holomorphic for $S < |z-a| < \infty$. The part $\sum_{n=0}^{\infty} c_n(z-a)^n$ containing non-negative powers is holomorphic for $0 \leq |z-a| < R$. Therefore these two parts of the series are both holomorphic on the *overlap region* $S < |z-a| < R$.

Classification of singularities

Suppose that $S = 0$ in Laurent's theorem, so that $f(z)$ is holomorphic on the *punctured disc* $D'(a; R) = \{z \in \mathbb{C} : 0 < |z-a| < R\}$ (where it may happen that $R = \infty$). Then $f(z)$ has an *isolated singularity* at $z = a$. These singularities can be classified into three categories, as follows.

1. If all the negative coefficients in the Laurent expansion vanish, $c_n = 0$ for all $n < 0$, then $f(z)$ can be made holomorphic at $z = a$ by setting $f(a) = c_0 = \lim_{z \to a} f(z)$. Such a singularity is termed *removable*.

2. If there is an integer $m > 0$ such that $c_{-m} \neq 0$ but $c_n = 0$ for $n < -m$, then $f(z)$ has a *pole of order* m at $z = a$. In this case $(z-a)^m f(z)$ is holomorphic at $z = a$. A function whose only singularities are poles is called *meromorphic*.

3. If neither of the above holds, then there are infinitely many nonzero negative Laurent coefficients: the principal part goes on for ever. In this case $f(z)$ has an *isolated essential singularity* at $z = a$.

The prototypical example of the third case is the function $\mathrm{e}^{1/z}$, which has an essential singularity at $z = 0$. The behaviour of $f(z)$ near a pole of order m may be singular, but the singularity is relatively tame because $(z-a)^m f(z)$ is locally holomorphic. However, the behaviour near an isolated essential singularity is truly pathalogical, as demonstrated by:

Picard's Theorem: *Suppose $f(z)$ has an isolated singularity at $z = a$ and let D_ϵ be a small neighbourhood of $z = a$ such that $f(z)$ is holomorphic in $D_\epsilon \setminus \{a\}$. Then, for every $\zeta \in \mathbb{C}$, with at most one exception (the so-called lacunary value), the equation $f(z) = \zeta$ has infinitely many roots z in D_ϵ.*

That is, the image of D_ϵ under $f(z)$ covers the *whole* complex plane (except for at most one point) *infinitely often*! For the particular example $f(z) = \mathrm{e}^{1/z}$, it is quite easy to see why this is true, and that the lacunary value in this case is zero.

The behaviour of $f(z)$ at infinity is classified according to the behaviour of $f(1/z)$ near $z = 0$; thus, for example, z has a pole of order 1 at infinity, while e^z has an essential singularity at infinity.

1.7 Cauchy's Residue Theorem

If $f(z)$ has an isolated singularity at $z = a$ and is otherwise holomorphic inside and on a contour Γ enclosing a, then (by uniform convergence)

$$\begin{aligned} \oint_\Gamma f(z)\,\mathrm{d}z &= \oint_\Gamma \sum_{n=-\infty}^{\infty} c_n (z-a)^n \,\mathrm{d}z \\ &= \sum_{n=-\infty}^{\infty} c_n \oint_\Gamma (z-a)^n \,\mathrm{d}z \\ &= 2\pi\mathrm{i} c_{-1}, \end{aligned} \tag{1.24}$$

as all the other integrals vanish. The coefficient

$$c_{-1} = \mathrm{res}[f(z); a] \tag{1.25}$$

is the called *residue* of $f(z)$ at $z = a$.

The result is easily generalised to the case when $f(z)$ has several isolated singularities inside Γ, and is known as:

Cauchy's Residue Theorem: *If $f(z)$ is holomorphic inside and on Γ with the exception of a finite number of isolated singularities at $z = a_j$ inside Γ, then*

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_j \mathrm{res}[f(z); a_j]. \tag{1.26}$$

Calculation of residues relies on a few variations on the theme of calculating local expansions. Apart from functions such as $\mathrm{e}^{1/z}$ for which we just calculate the power series, we may often have functions with poles, in the form

$$f(z) = \frac{g(z)}{h(z)} \tag{1.27}$$

where both $g(z)$ and $h(z)$ are holomorphic at $z = a$ and $h(a) = 0$, $g(a) \neq 0$. The order of the pole then depends on the order of the zero of $h(z)$ at $z = a$. If $h(z) = z - a$ the pole is a simple one and the residue is $g(a)$, and if $h(z) = (z-a)^n$, the Taylor expansion of $g(z)$ shows that the residue is $g^{(n-1)}(a)/(n-1)!$. If $h(a) = 0$ but $h'(a) \neq 0$, expanding $h(z) = (z-a)h'(a) + \cdots$ shows that the residue is $g(a)/h'(a)$; and so on.

1.8 Multifunctions

A function $f(z)$ has a *branch point* at $z = a$ if, on taking a circuit round a, the final value of $f(z)$ is not equal to the original one. It is possible for a branch point to be at infinity: we say that $f(z)$ has a branch point at infinity if $f(1/z)$ has a branch point at the origin.

The most basic example is $\log z$, defined as

$$\log z = \log|z| + \mathrm{i}\arg z = \log r + \mathrm{i}\theta, \tag{1.28}$$

where $|z| = r$ and $\arg z = \theta$ are the *modulus* and *argument* of z, defined such that

$$z = r\mathrm{e}^{\mathrm{i}\theta}. \tag{1.29}$$

It is clear that (1.29) only defines θ up to an arbitrary integer multiple of 2π, and it is this ambiguity that leads to the multivaluedness of $\log z$. If we perform a circuit of the origin by setting $z = \epsilon\mathrm{e}^{\mathrm{i}t}$ and then letting t increase from 0 to 2π, then the value of $\log z$ does not return to its initial value $\log\epsilon$ but increases by $2\pi\mathrm{i}$. Thus there is a branch point at $z = 0$. There is also a branch point at infinity, since $\log(1/z)$ has a branch point at $z = 0$.

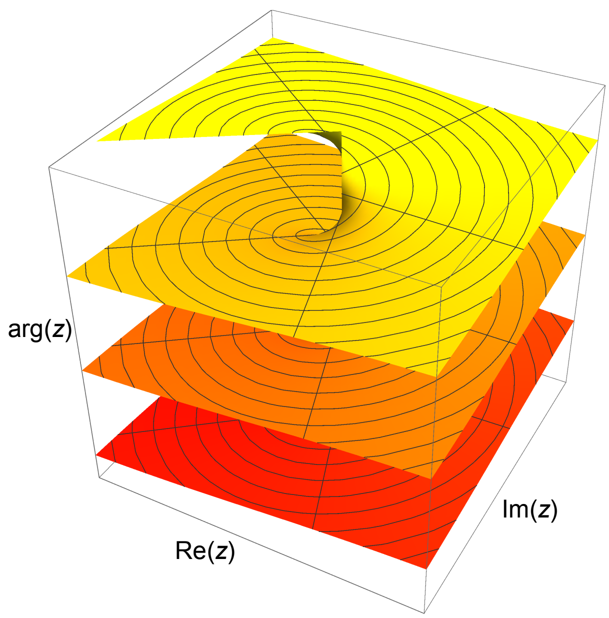

Figure 1.6: The Riemann surface for $\arg z$.

There are two solutions to this difficulty. One is to construct a *Riemann surface*, consisting of all possible values of the multifunction at each point of $\mathbb{C}$. For example, given $z \neq 0$, the argument θ of z takes a countably infinite set of values (all differing by integer multiples of 2π), resulting in the surface shown in Figure 1.6. Thus the Riemann surface for $\log z$ defined in this way resembles a multi-storey carpark. At each point on this surface, apart from the branch point $z = 0$ itself, $\log z$ defines a locally holomorphic function, but because of the mutiple layers of the surface, the value of $\log z$ for a given value of z is ambiguous.

The second solution is to restrict the domain of definition of the function so that the problematic circuits are forbidden. This is achieved by introducing *branch cuts*, joining the branch points, across which contours may not pass. Then it is possible to define single-valued *branches* of the multifunction. For example, we can define a single-valued branch of $\log z$ by drawing a branch cut that connects the branch points $z = 0$ and $z = \infty$. Obviously there is no unique way of doing so, but the most popular choice is to place the branch cut along the negative real axis, as shown in Figure 1.7. Thus we define a countable set of branches of $\log z$ corresponding to each integer k, with

$$\log z = \log r + \mathrm{i}\theta + 2\pi k\mathrm{i}, \tag{1.30}$$

and θ now restricted to lie in the range $-\pi < \theta \leq \pi$ (so that the cut cannot be crossed). Each choice of k results in a single-valued function which is holomorphic on the cut plane.

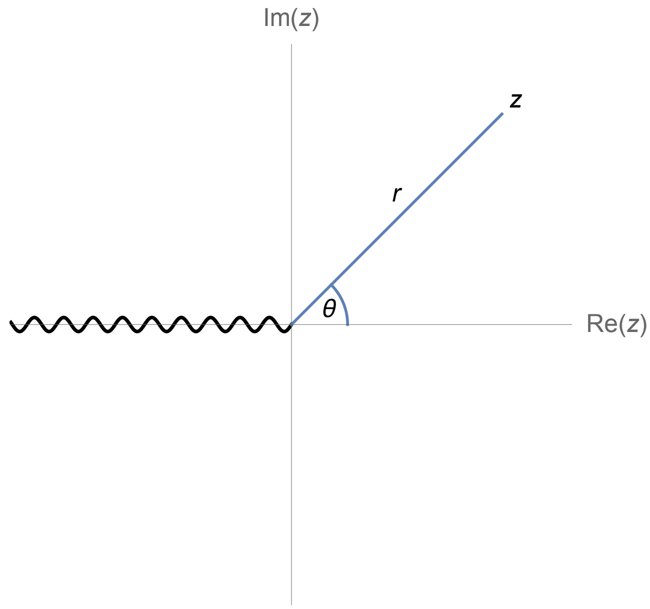

Figure 1.7: The complex plane with a branch cut along the negative real axis.

The price we pay is that the resulting function is discontinuous across the branch cut. The particular branch with $k = 0$ is often called the *principal branch* of $\log z$, and denoted by $\operatorname{Log} z$; the corresponding principal branch of $\arg z = \theta \in (-\pi, \pi]$ is denoted by $\operatorname{Arg} z$.

One other instructive example is the multifunction $\left(z^2-1\right)^{1/2}$, which can be defined by

$$\begin{aligned}\left(z^2-1\right)^{1/2} &= (z-1)^{1/2}(z+1)^{1/2} \\ &= (r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2},\end{aligned} \tag{1.31}$$

where

$$r_1 = |z-1|, \qquad r_2 = |z+1|, \qquad \theta_1 = \arg(z-1), \qquad \theta_2 = \arg(z+1). \tag{1.32}$$

One can easily check that $\left(z^2-1\right)^{1/2}$ has branch points at $z = \pm 1$. Choosing a particular branch of the multifunction corresponds to uniquely defining the two angles θ_1 and θ_2, and there are two canonical choices, the branch cuts for which are illustrated in Figure 1.8.

First suppose we define θ_1 and θ_2 such that

$$-\pi < \theta_1 \le \pi, \qquad\qquad -\pi < \theta_2 \le \pi. \tag{1.33}$$

The corresponding function defined by (1.31) has a branch cut that connects $z = -1$ to $z = 1$ along the line segment $[-1, 1]$ on the real axis. Just above the cut, we have $\theta_1 = \pi$, $\theta_2 = 0$ and therefore $\left(z^2-1\right)^{1/2} = \mathrm{i}\left|z^2-1\right|$. Just below the cut, we have $\theta_1 = -\pi$, $\theta_2 = 0$ and therefore $\left(z^2-1\right)^{1/2} = -\mathrm{i}\left|z^2-1\right|$. Thus the function is discontinuous and changes sign across the branch cut. Across the negative real axis with $\operatorname{Re}(z) < -1$, both θ_1 and θ_2 jump by 2π, so that the argument $(\theta_1+\theta_2)/2$ jumps by 2π, returning the same value of $\left(z^2-1\right)^{1/2}$. This explains why there is no discontinuity across the negative real axis where $\operatorname{Re}(z) < -1$.

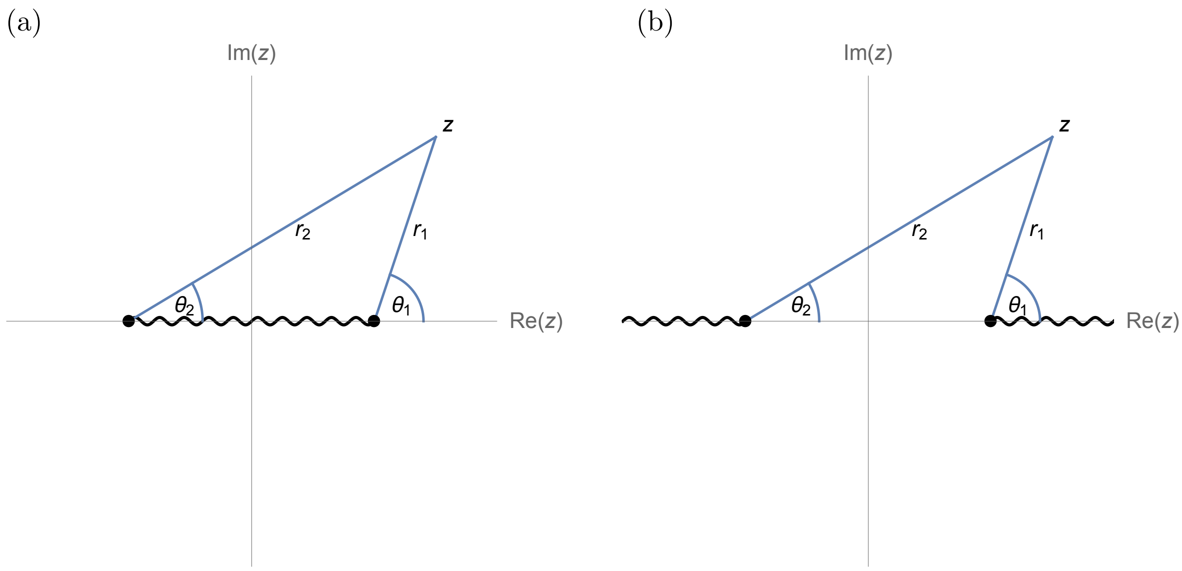

Figure 1.8: Two possible branch cuts for the multifunction $\left(z^2-1\right)^{1/2}$.

The second popular definition of $\left(z^2-1\right)^{1/2}$ occurs when we choose θ_1 and θ_2 in the ranges

$$-\pi < \theta_1 \le \pi, \qquad 0 < \theta_2 \le 2\pi. \tag{1.34}$$

In this case, the branch cut extends along the real axis on the intervals $(-\infty,-1]$ and $[1,\infty)$. There is not a branch point at infinity, so we should not really think of there being two branch cuts that extend to infinity, but of a single cut that connects -1 to 1 through infinity.

1.9 Evaluation of integrals

There is a collection of standard contours which are used to evaluate standard integrals. Cauchy's integral theorems apply to integration around closed contours. Often this involves closing a contour on a suitable return path and then proving that the additional contribution to the integral tends to zero as the return path tends to infinity.

Example 1

The substitution $\mathrm{e}^{\mathrm{i}\theta}=z$ transforms integrals of rational functions of $\sin\theta$, $\cos\theta$ to integrals of rational functions of z. For example, for the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\sin\theta},$$

with $0<b<a$, the substitution $\mathrm{e}^{\mathrm{i}\theta}=z$ results in an integral around the unit circle, namely

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\sin\theta} = \oint_{|z|=1} \frac{2\,\mathrm{d}z}{bz^2+2\mathrm{i}az-b}. \tag{1.35}$$

The integrand has poles on the negative imaginary axis, only one of which is inside the unit disc, as illustrated in Figure 1.9, and the result is easily found using Cauchy's Residue Theorem.

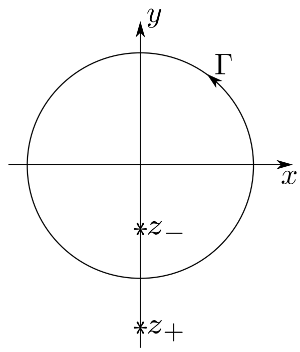

Figure 1.9: The integrand of the integral (1.35) has two simple poles at $z = z_{\pm}$.

Example 2

Consider the integral of a rational function of x, of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, \mathrm{d}x,$$

where $P(x)$ and $Q(x)$ are polynomials with $\deg(Q) \geq \deg(P) + 2$ and where Q has no real roots. Here we first take the integral from $x = -R$ to $x = R$ and close with a large semicircle in the upper half-plane, then let $R \to \infty$, as shown in Figure 1.10. Our assumption about the degrees of P and Q ensures that the contribution from the semicircle tends to zero as $R \to \infty$. Then we just need to apply Cauchy's Residue Theorem and sum up the contributions from all the residues in the upper half-plane. (Closing the contour in the lower half-plane would give the same answer, but we would have to remember an extra minus sign because Γ is then taken clockwise.)

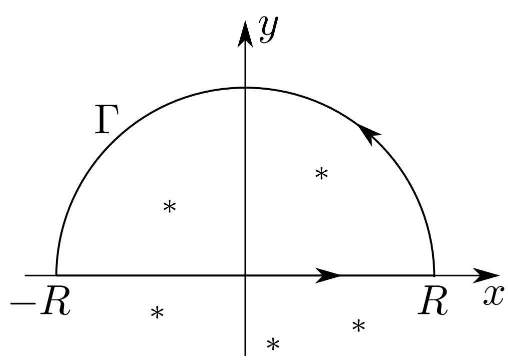

Figure 1.10: Closing the contour in the upper half-plane.

Example 3

For integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)\cos x}{Q(x)} \, \mathrm{d}x \qquad \text{or} \qquad \int_{-\infty}^{\infty} \frac{P(x)\sin x}{Q(x)} \, \mathrm{d}x$$

we integrate

$$\frac{e^{iz}P(z)}{Q(z)}$$

round a semicircular contour closed in the *upper* half-plane. We must use the upper half-plane to ensure that there is no contribution from the semicircle, because

$$e^{iz} = e^{ix-y}$$

decays as $y \to +\infty$ but blows up as $y \to -\infty$. In this case, the integral exists provided $\deg(Q) \geq \deg(P) + 1$. For the limiting case where $\deg(Q) = \deg(P) + 1$, the contribution from the semicircle can be bounded by using **Jordan's result** that

$$\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \to 0 \quad \text{as } R \to \infty. \tag{1.36}$$

Example 4

Sometimes it is possible to construct a contour Γ such that the integral along one segment of Γ is a constant multiple of the integral along another (together with vanishing contributions from the remaining segments). For example, to evaluate

$$\int_0^\infty \frac{dx}{1+x^{2n}},$$

take Γ to run from 0 or R along the real axis, then via a circular arc to return along the ray $\arg z = \pi/n$ (on which $1 + z^{2n}$ is the same as on the real axis), picking up the residue from the pole at $z = e^{i\pi/2n}$, as shown in Figure 1.11 (left).

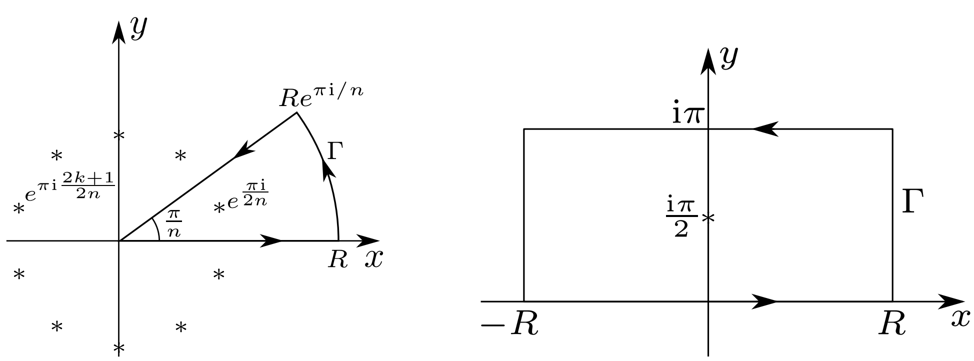

Figure 1.11: Closed contours for $f(z) = 1/(1 + z^{2n})$ (left) and $f(z) = \cos(z)/\cosh(z)$ (right).

Or, to evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\,dx,$$

integrate $\cos z/\cosh z$ round a rectangular contour with corners at $\pm R$, $\pm R + i\pi$, as shown in Figure 1.11 (right), enclosing the pole at $z = i\pi/2$.

Example 5

Integrals such as

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x,$$

can be evaluated by integrating $\mathrm{e}^{\mathrm{i}z}/z$ round a contour consisting of the real axis with a small semicircular indentation at the origin, above the real axis, plus a large semicircle, as shown in Figure 1.12. The small indentation is necessary to avoid the singularity at $z = 0$, and contributes $-\pi\mathrm{i}$ times the residue at the origin. The minus sign is because the semicircle is taken clockwise, and the factor of $\pi\mathrm{i}$ instead of $2\pi\mathrm{i}$ comes from having only half a circle, so we only pick up half the residue.

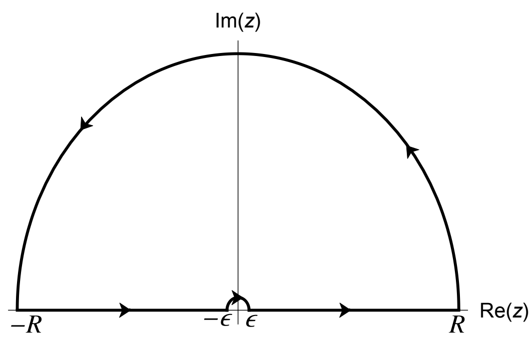

Figure 1.12: Integration contour for the integrand $\mathrm{e}^{\mathrm{i}z}/z$.

Example 6

A 'keyhole' contour is necessary for integrals involving branch cuts, for example

$$I_1 = \int_0^\infty \frac{\log x\,\mathrm{d}x}{(x+a)(x+b)} \quad (a, b > 0) \qquad \text{or} \quad I_2 = \int_0^\infty \frac{\log^2 x\,\mathrm{d}x}{1+x^2}. \tag{1.37}$$

For I_1, integrate $(\log z - \pi\mathrm{i})^2/(z+a)(z+b)$ (with the branch cut chosen along the positive real axis) round the contour shown in Figure 1.13 and exploit the different values of the log on either side of the cut. The same contour also works for I_2 but with the integrand $(\log z - \pi\mathrm{i})^3/(1+z^2)$.

1.10 Fourier and Laplace transforms

The **Fourier transform** of a real function $f(x)$ is defined by

$$\bar{f}(k) \equiv \mathcal{F}[f] = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x, \tag{1.38}$$

and the inverse is

$$f(x) \equiv \mathcal{F}^{-1}\left[\bar{f}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{1.39}$$

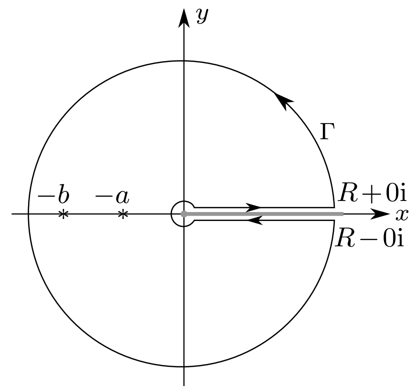

Figure 1.13: Closed contour for the integrals in Example 6. The branch cut is marked as a grey line; note that $\log z - \pi\mathrm{i} = \log(x) \mp \pi\mathrm{i}$ for $z = x \pm 0\mathrm{i}$, $x > 0$.

Note that there are various different conventions in defining the Fourier transform, but this is the version that will be used in this course. In practice, inversion is usually accomplished by contour integration in the complex k-plane.

Integration by parts shows that

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = -\mathrm{i}k\hat{f}(k), \tag{1.40}$$

so that the Fourier transform converts differential operators into algebraic operators. On the other hand, differentiation under the integral sign leads to

$$\mathcal{F}[xf(x)] = -\mathrm{i}\frac{\mathrm{d}\bar{f}}{\mathrm{d}k}. \tag{1.41}$$

The **Laplace transform** operates on functions defined on the positive real axis:

$$\hat{f}(p) \equiv \mathcal{L}[f] = \int_0^\infty f(x)\mathrm{e}^{-px}\,\mathrm{d}x. \tag{1.42}$$

If $f(x)\mathrm{e}^{-\gamma x}$ is integrable (so that $|f(x)|$ grows no faster than $\mathrm{e}^{\gamma x}$ as $x \to \infty$), then $\hat{f}(p)$ exists for $\mathrm{Re}\,p \geq \gamma$ and is holomorphic in p for $\mathrm{Re}\,p > \gamma$. It can usually be analytically continued into the rest of the complex p-plane, although singularities inevitably occur. The Laplace inversion formula is

$$f(x) \equiv \mathcal{L}^{-1}\left[\hat{f}\right] = \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \hat{f}(p)\mathrm{e}^{px}\,\mathrm{d}p. \tag{1.43}$$

The contour is usually (but not always) completed in the left-hand half-plane and in many problems the solution is given by a sum of residues from the interior of the completed contour. If a branch cut is present in $\hat{f}(p)$ then some kind of keyhole contour is required.

1.11 Conformal mapping

We can view a holomorphic function $f(z)$ as a mapping from a point z in the complex plane to a new point $\zeta = f(z)$. We then ask the question: given a domain in the z-plane, what is the image of that domain in the ζ-plane under the mapping $z \mapsto \zeta = f(z)$?

Basic properties

Suppose $f(z)$ is holomorphic at $z = a$. Then, by Taylor's theorem, for z near a we have

$$\zeta = f(z) = f(a) + f'(a)(z - a) + \cdots . \tag{1.44}$$

If $f'(a) \neq 0$, this shows that a small neighbourhood of $z = a$ is translated, with $z = a$ going to the point $\zeta = f(a)$, and then rotated (by an angle $\arg f'(a)$) and scaled (by a factor $|f'(a)|$), as illustrated in Figure 1.14.

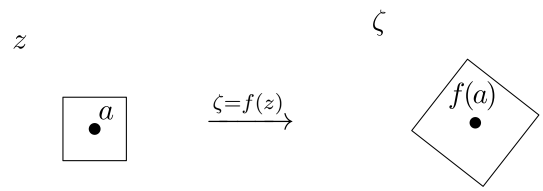

Figure 1.14: A small region of the z-plane is translated, rotated and scaled under the mapping $z \mapsto \zeta = f(z)$. (Here and henceforth, the z- and ζ-planes are just labelled "z" and "ζ".)

If two curves meet at $z = a$ and the angle between them is α, it follows from the local linearity that the angle between their images is also α (and has the same sense), as shown in Figure 1.15. Maps with this property are called *conformal*, and we have shown that a holomorphic function is a conformal map at all points where its derivative does not vanish.

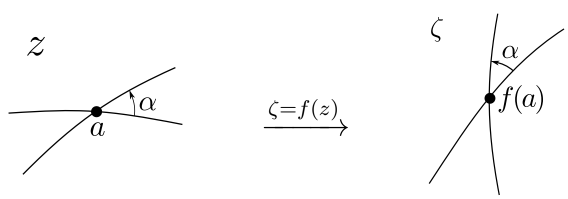

Figure 1.15: The angle (including sense) between two curves is preserved by the mapping $z \mapsto \zeta = f(z)$.

A conformal map is *locally* one-to-one, being a small perturbation of a linear map, but this is only a local statement. When we look at the image of a domain D under $f(z)$, the map may not be *globally* one-to-one, even if $f'(z)$ does not vanish in D. Figure 1.16 illustrates the kind of problem that may arise.

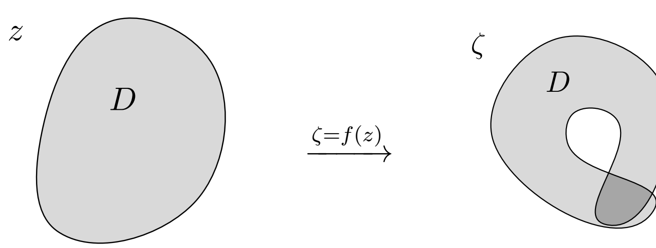

Figure 1.16: A conformal map need not be globally one-to-one.

The composition of two conformal maps is itself conformal. We can use this in building up complicated maps from simple ones.

Behaviour near a critical point

We have shown that the mapping $z \mapsto \zeta = f(z)$ is conformal provided $f'(z) \neq 0$. However, many useful mapping functions have isolated *critical points* where f' is zero. Consider a point a such that $f'(a) = 0$ and $f''(a) \neq 0$. If we write $z = a + r\mathrm{e}^{\mathrm{i}\theta}$, then

$$\begin{aligned} f(z) &= f(a) + \frac{1}{2} f''(a)(z-a)^2 + O\left((z-a)^3\right) \\ &= f(a) + \frac{1}{2} f''(a) r^2 \mathrm{e}^{2\mathrm{i}\theta} + O(r^3). \end{aligned} \tag{1.45}$$

The image of a small neighbourhood of $z = a$ now covers a small neighbourhood of $f(a)$ *twice*, as shown in Figure 1.17, and the angle between two curves meeting at a is doubled.

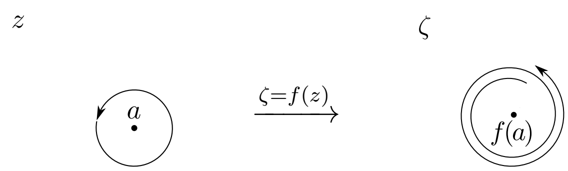

Figure 1.17: If $f'(0) = 0$ and $f''(a) \neq 0$ then angles are locally doubled.

We see that if a map $f(z)$ has a critical point at a point $z = a$ in a region D, it is neither conformal at that point, nor locally one-to-one near it. The only hope of constructing a one-to-one map using $f(z)$ is if a lies on the boundary of D. A very simple example is $f(z) = z^2$ acting on the right-hand half-plane $\operatorname{Re} z > 0$. The image of an interior point $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi/2 < \theta < \pi/2$, is $\zeta = r^2\mathrm{e}^{2\mathrm{i}\theta}$, and we see that $-\pi < \arg\zeta < \pi$, so the map is one-to-one (as predicted by the doubling of the angles). The image of the half-plane $\operatorname{Re} z > 0$ is the entire ζ-plane, minus a slit along the negative real axis, as shown in Figure 1.18.

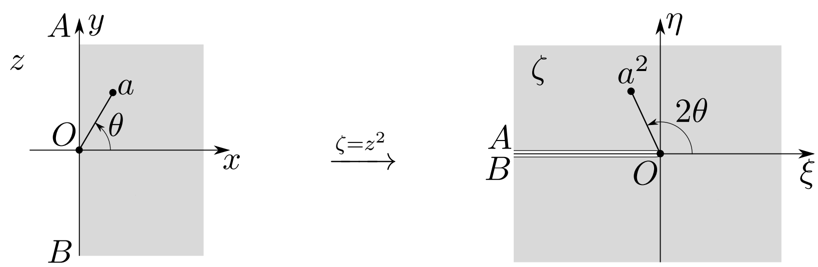

Figure 1.18: The mapping $z \mapsto \zeta = z^2$ maps the half-plane $\operatorname{Re} z > 0$ to the entire ζ-plane, minus a slit along the negative real axis

The Riemann mapping theorem

It is natural to ask what domains we can map to what. The answer is that virtually any simply-connected region of the complex plane can be mapped to virtually any other such region, because of:

The Riemann Mapping Theorem. *Any simply connected domain D, with the sole exception of $\mathbb{C}$ itself, can be mapped conformally onto the unit disc $|\zeta| < 1$. There are three free real parameters in the map.*

We omit the proof, which is very technical and not constructive, and instead just make some comments on the theorem.

1. The reason that there is no map from $\mathbb{C}$ itself to the unit disc is that if there were one, the mapping function would be entire and its modulus would be bounded by 1; by Liouville's theorem, the function would have to be constant (and hence not conformal).

2. The three real degrees of freedom might look mysterious. As we shall see, they arise because there is precisely a three-parameter family of maps from the unit disc to itself.

3. If D is bounded by a simple closed curve (which may pass through the point at infinity), we may use the three degrees of freedom to specify the images of three boundary points, and the map is then uniquely determined. Alternatively, we may specify the image of one interior point of D, and the image of a direction at that point.

4. It is conventional to map to the unit disc. However, any other 'canonical domain' would do, since we can map the disc onto it and then use the composition of maps. The upper half-plane is also often used as a canonical domain, and then the three boundary points in Riemann's theorem are usually taken to be 0, 1 and ∞.

5. The theorem says that the map is conformal in the interior of D (so its inverse is conformal in $|\zeta| < 1$), but it might not be conformal on the boundary of D, where singularities are needed to smooth out corners and cusps.

6. Because conformal maps preserve angles, including their orientation, if we go round ∂D in a particular sense (say, anticlockwise), then the image of ∂D is traversed in the same sense, as illustrated in Figure 1.19.

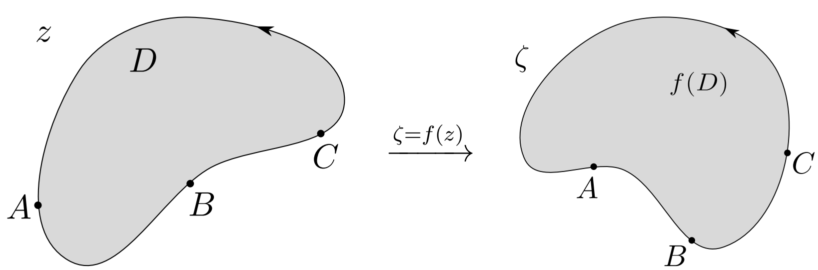

Figure 1.19: The orientation of the boundary is preserved by a conformal mapping, so that the images of the points A, B, C occur in the same order.

Standard maps

Let us look at some standard maps. In the accompanying examples, we see how to construct complicated maps from these building blocks.

Bilinear (Möbius) maps

The Möbius transformation is

$$\zeta = f(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}, \tag{1.46}$$

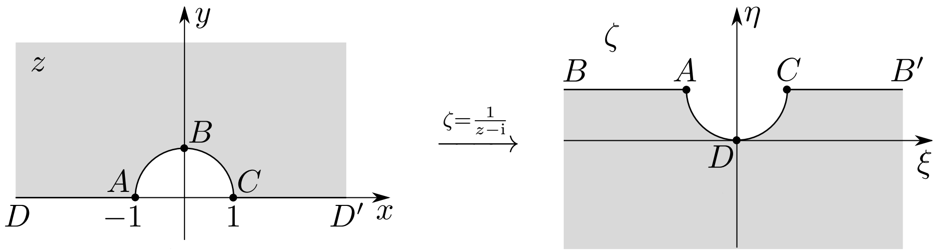

Figure 1.20: The effect of the Möbius mapping $z \mapsto \zeta = 1/(z - \mathrm{i})$ on a half-plane with a semi-circle removed.

where $ad - bc \neq 0$ (otherwise, the map is constant) and, to avoid trivial translations, rotations and scalings, $c \neq 0$. The Möbius transformation is conformal at all points except the location of its pole, and is a one-to-one map from the *extended complex plane* $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

It is a well known fact that Möbius transformations take circles and straight lines into circles and straight lines. (The jargon "circlines" is sometimes used for circles and straight lines; a straight line can be viewed as a circle with its centre at infinity.) It is not usually necessary to do detailed calculations when using Möbius maps on a domain bounded by straight lines and circles; it is enough to know that the boundary maps to a combination of straight lines and circles, and to know the angles where they meet, since these are also preserved by the map.

Example. The domain D consists of the upper half-plane $y > 0$ with its intersection with the closed unit disc removed (thus, it has a semicircular indentation centred on the origin). Find the image of the domain under the map $\zeta = 1/(z - \mathrm{i})$, indicating the images of significant boundary points.

Solution. The image boundary is made up of straight lines and circles. The points $z = \pm 1$ map to $\zeta = 1/(\pm 1 - \mathrm{i}) = (\pm 1 + \mathrm{i})/2$, respectively, $z = \infty$ maps to $\zeta = 0$, and $z = \mathrm{i}$ maps to $\zeta = \infty$. Therefore, the unit circle is mapped to the line $\eta = 1/2$ and the x-axis is mapped to the circle $|\zeta - \mathrm{i}/2| = 1/2$.

The image of the upper half-plane with the closed unit disc removed is then the lower half-plane $\eta < 1/2$ with the closed disc $|\zeta - \mathrm{i}/2| \leq 1$ removed, as shown in Figure 1.20. Note the order of the points marked A, B, C, D is preserved by the mapping, and that the shaded region remains on the left-hand side as we follow the boundary in the order $A \to B \to C \to D$.

Example. Find a map from the domain of the previous example to the quarter plane $\xi > 0$, $\eta > 0$.

Solution. Start by looking at the angles. We have two right angles at $z = \pm 1$, which will be preserved by a Möbius transformation. We shift one right-angle to $\zeta = 0$ and the other to $\zeta = \infty$ with the mapping

$$\zeta = \frac{z - 1}{z + 1}. \tag{1.47}$$

Then $z = 1$ is mapped to $\zeta = 0$, $z = \infty$ is mapped to $\zeta = 1$, $z = -1$ is mapped to $\zeta = \infty$, and $z = \mathrm{i}$ is mapped to $\zeta = \mathrm{i}$. This is enough information to deduce that the boundary gets mapped to the coordinate axes, as shown in Figure 1.21.

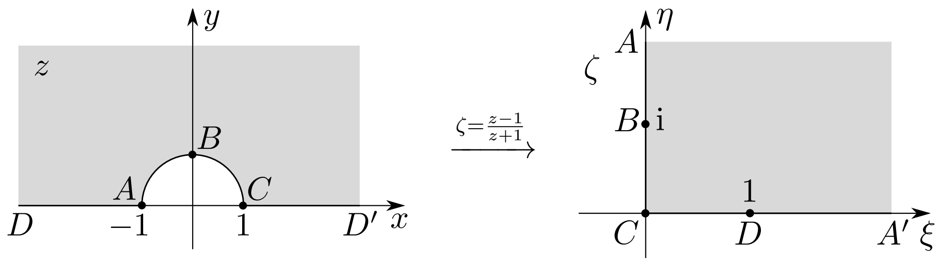

Figure 1.21: Mapping from a half-plane with a semi-circle removed to a quarter-plane.

Example. Map the upper half-plane $y > 0$ to the unit disc $|\zeta| < 1$.

Solution. The upper half-plane is described by the inequality

$$\left|\frac{z - \mathrm{i}}{z + \mathrm{i}}\right| < 1, \tag{1.48}$$

i.e. the set of points closer to i than to −i. This is exactly what we need: if

$$\zeta = \frac{z - \mathrm{i}}{z + \mathrm{i}}, \tag{1.49}$$

then the upper half-plane is mapped to $|\zeta| < 1$.

Example. If $|w| < 1$ and $\theta \in \mathbb{R}$, show that the map

$$\zeta = \mathrm{e}^{\mathrm{i}\theta}\,\frac{z - w}{1 - \overline{w}z} \tag{1.50}$$

maps the unit disc one-to-one to itself.

Solution. The map (1.50) is a Möbius transformation, with its pole at $z = 1/\overline{w}$ outside the unit disc, so is automatically one-to-one. A direct calculation reveals that

$$1 - |\zeta|^2 \equiv \left[\frac{1 - |w|^2}{|1 - \overline{w}z|^2}\right]\left(1 - |z|^2\right). \tag{1.51}$$

Since $|w| < 1$, the term in square brackets is strictly positive, and it follows that $|\zeta|^2 < 1$ if and only if $|z|^2 < 1$.

It can be shown that Möbius maps of the form (1.50) are the *only* one-to-one maps of the unit disc to itself. The two real parameters needed to define w, and the single real angle of rotation θ, are the three free parameters in the Riemann mapping theorem: having mapped a domain onto the unit disc, we can subsequently apply any map of the form (1.50) while preserving the image as the unit disc.

Example. Map the upper half plane to itself, permuting the points 0, 1 and ∞.

Solution. Because the map preserves the orientation of the interior of the domain vis a vis its boundary, only the even permutations, in which $(0, 1, \infty)$ are mapped to $(1, \infty, 0)$ or $(\infty, 0, 1)$ are possible. We need a Möbius map with real coefficients, so that the real axis maps to itself. First take $(0, 1, \infty)$ to $(1, \infty, 0)$. This means that the map has its pole at $z = 1$, so the denominator is $z - 1$ (or a multiple thereof); infinity maps to zero, so the numerator must be a constant (so the map looks like constant$/z$ at infinity; finally, the remaining constant is fixed by the requirement that 0 maps to 1, and we have

$$\zeta = \frac{1}{1 - z}. \tag{1.52}$$

The map taking $(0, 1, \infty)$ to $(\infty, 0, 1)$ is likewise found to be

$$\zeta = \frac{z - 1}{z}; \tag{1.53}$$

it sends $z = 0$ to infinity, $z = 1$ to 0, and is asymptotic to 1 at infinity, as required.

Powers of z

The powers of z are of most interest when the boundary of D contains a corner at the origin. If D is a sector $0 < \arg z < \alpha$, then the image of this wedge under the map $\zeta = z^n$ is the sector $0 < \arg \zeta < n\alpha$, as shown in Figure 1.22, provided $n\alpha < 2\pi$ so that the map is conformal. This multiplying of angles by n is often useful in getting rid of corners in ∂D, as the next two examples show. Note that there is no need for n to be an integer; if we want to multiply an angle by 3/2 we choose $n = 3/2$, and if we want to halve an angle we choose $n = 1/2$, and so on.

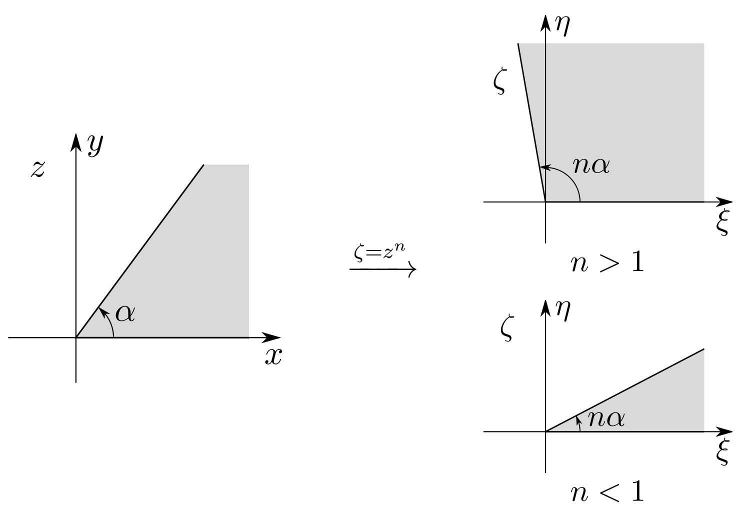

Figure 1.22: The action of the mapping $\zeta = z^n$ on a sector.

Example. The domain D consists of the sector $-\pi/6 < \arg z < \pi/6$ of the unit disc (a slice of pizza). Map it to a semicircle of radius 2.

Solution. We need to get rid of the angle of $\pi/3$ at the origin. The map $\zeta_1 = z^3$ does exactly this; however it takes the unit circle to the unit circle. Hence the required map is $\zeta = 2\zeta_1 = 2z^3$, as shown in Figure 1.23.

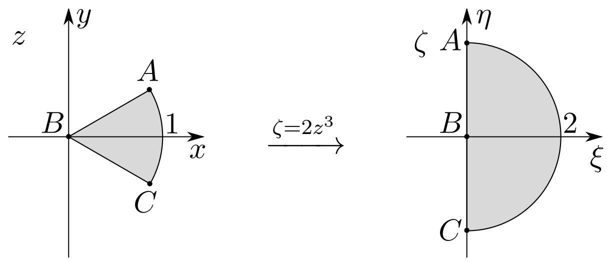

Figure 1.23: Mapping a slice of pizza to a semi-circle of radius 2.

Example. The domain D consists of the unit disc with the sector $\frac{5\pi}{6} < \arg z < \frac{7\pi}{6}$ removed, so it looks like a partly eaten pizza. Find a map from D to the upper half-plane.

Solution. The boundary of D has three corners, with angle $5\pi/3$ at the origin, and two angles of $\pi/2$ on the unit circle. We need to get rid of all of these. We do the mapping in 3 stages.

1. First put
$$\zeta_1 = z^{3/5}. \tag{1.54a}$$
This gets us to a semicircle $-\pi/2 < \arg \zeta_1 < \pi/2$, $0 < |\zeta_1| < 1$.

2. Map the semi-circle in the ζ_1-plane to a quarter plane in the ζ_2-plane using a Möbius transformation. We have two right angles at $\zeta_1 = \pm\mathrm{i}$. We choose the mapping such that $\zeta_1 = -\mathrm{i}$ is mapped to $\zeta_2 = 0$ and $\zeta_1 = \mathrm{i}$ is mapped to $\zeta_2 = \infty$, so ζ_2 is a multiple of $(\zeta_1 + \mathrm{i})/(\zeta_1 - \mathrm{i})$. If we also specify that the image of $\zeta_1 = 0$ is $\zeta_2 = 1$ we find
$$\zeta_2 = -\frac{\zeta_1 + \mathrm{i}}{\zeta_1 - \mathrm{i}}. \tag{1.54b}$$
which maps the semi-circle to the fourth quadrant in the ζ_2-plane.

3. The map
$$\zeta = -\zeta_2^2 \tag{1.54c}$$
will map the quarter plane $-\pi/2 < \arg \zeta_2 < 0$ to the upper half-plane $\eta > 0$.

The combined map is

$$\zeta = -\zeta_2^2 = -\left(-\frac{\zeta_1 + \mathrm{i}}{\zeta_1 - \mathrm{i}}\right)^2 = -\left(\frac{z^{3/5} + \mathrm{i}}{z^{3/5} - \mathrm{i}}\right)^2, \tag{1.55}$$

and the sequence of transformations is depicted in Figure 1.24.

Example. Map the semicircle $|z| < 1$, $0 < \arg z < \pi$, to the upper half-plane.

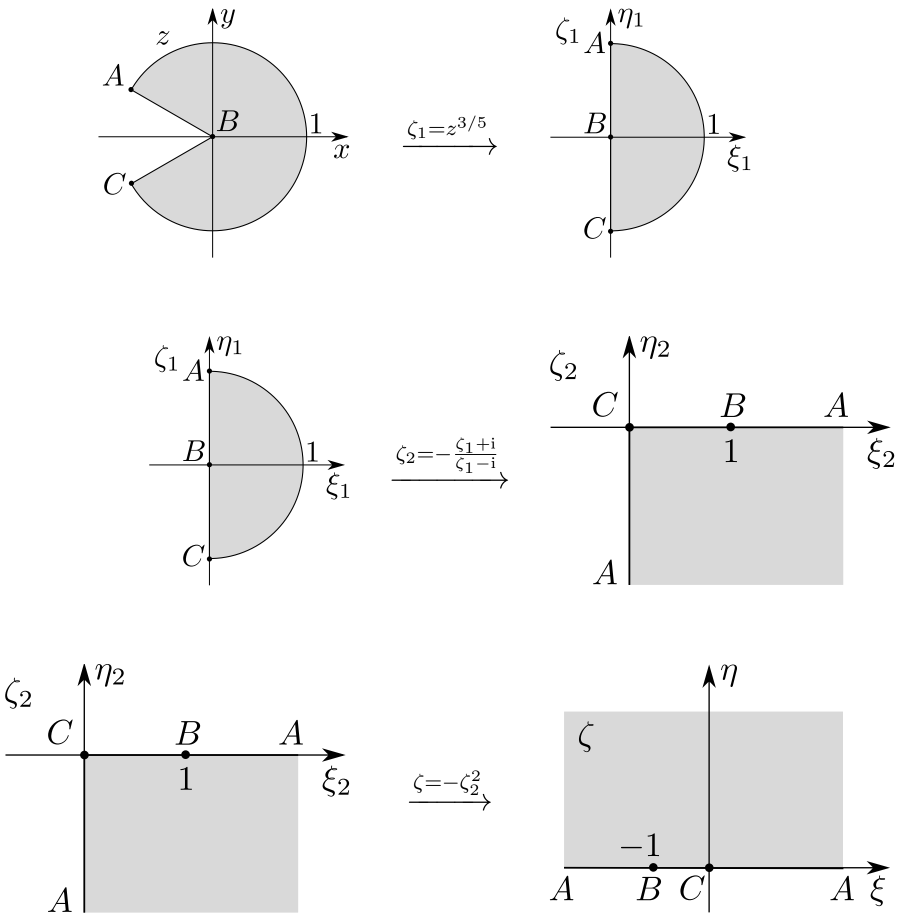

Figure 1.24: The sequence of maps from a pacman shape to the upper half-plane.

Solution. First we get rid of the semicircle by setting

$$\zeta_1 = \frac{1}{z+1}. \tag{1.56}$$

This inversion with respect to the left-hand corner point takes the semicircle to the quarter plane $\xi_1 > 1/2$, $\eta_1 < 0$. The quarter plane is mapped to the upper half plane by the transformation

$$\zeta = -(2\zeta_1 - 1)^2 = -\left(\frac{z-1}{z+1}\right)^2. \tag{1.57}$$

The succession of maps is depicted in Figure 1.25. Note that we could have made this a bit slicker by starting with the map

$$\zeta_1 = \frac{z-1}{z+1}, \tag{1.58}$$

which sends the corners of the semicircle to 0 and ∞, making the subsequent squaring easier.

Exponential and logarithm

Now consider the exponential function and its inverse, the logarithm. As $\mathrm{e}^{z+2\pi\mathrm{i}} \equiv \mathrm{e}^z$, the image of any horizontal strip of width 2π is repeated infinitely often, once for each of the

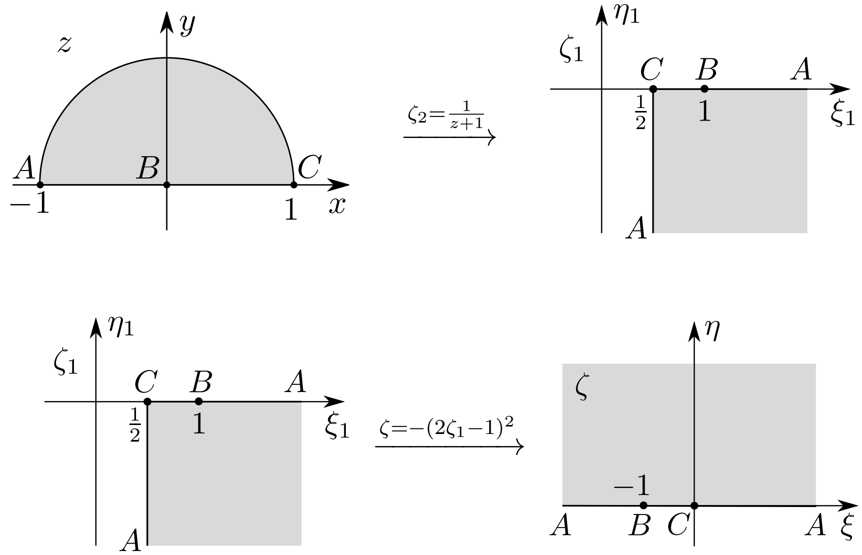

Figure 1.25: Mapping a semicircle to the upper half-plane.

strips obtained by shifting the original one vertically by 2π. Furthermore, as

$$\zeta = \mathrm{e}^z = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} \tag{1.59}$$

when $z = x + \mathrm{i}y$, the image of the 'baseline' strip $-\pi < y < \pi$, $-\infty < x < \infty$ is the whole plane with the negative real (ξ) axis removed. If we include the image of the line $y = -\pi$ as well, then the image of the strip $-\pi \le y < \pi$ is $\mathbb{C}\backslash\{0\}$; thus the image of the whole complex plane is the whole plane (minus the origin) covered infinitely often. This is an example of Picard's theorem in action: the function e^z has an essential singularity at infinity, and its lacunary value is 0.

Horizontal lines y = constant map onto rays $\arg\zeta = y$, while vertical line segments x = constant, $-\pi \le y < \pi$ map to circles of radius e^x. In particular, the imaginary axis is mapped to the unit circle. Thus the exponential map generates plane polar coordinates from a rectangular Cartesian grid.

The inverse map, the logarithm, is defined on the whole plane minus a cut from infinity to the origin. It takes the cut plane to a strip parallel to the ξ axis and of width 2π; if the cut is along $\arg z = \alpha$, then the strip is $\alpha - 2\pi < \eta < \alpha$, $-\infty < \xi < \infty$. Since

$$\zeta = \xi + \mathrm{i}\eta = \log z = \log|z| + \mathrm{i}\arg z, \tag{1.60}$$

circles $|z|$ = constant map to lines ξ = constant, and rays $\arg z$ = constant map to lines η = constant.

The effects of the exponential and logarithmic maps are illustrated in Figure 1.26. The exponential map is useful to open out a strip or half strip, and the log does the reverse. Note that neither exp nor log has a vanishing derivative.

Example. Map the half-strip $-\pi/2 < y < \pi/2$, $0 < x < \infty$ onto the interior of a semicircle, with the point at infinity mapping to the origin and the short side of the strip to the semicircle.

Solution. If we try to use the exponential function directly, we shall send the point at infinity to infinity. Instead, put $\zeta = \mathrm{e}^{-z}$; the result is illustrated in Figure 1.27.

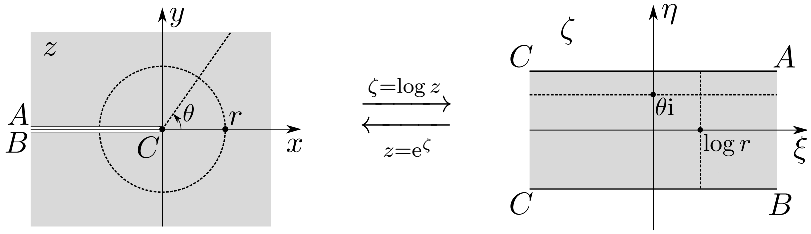

Figure 1.26: The effects of the exponential and logarithmic maps.

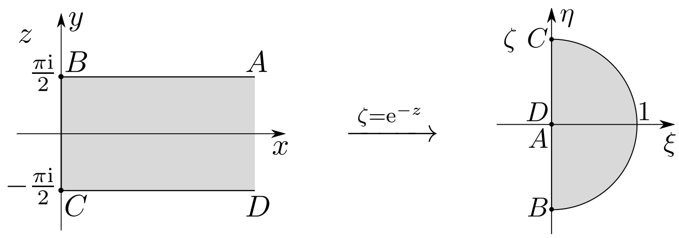

Figure 1.27: Mapping a half-strip to a semi-circle.

Example. The domain D consists of the upper half-plane with the upper half of the closed unit disc removed, as shown in the left-hand diagram in Figure 1.20. Map it to a half-strip.

Solution. We'll use the logarithmic map $\zeta = \log z$. The boundary of D consists of segments of rays and a segment of a circle centred at the origin. The rays map to horizontal lines and the semicircle to a segment of the imaginary axis. The image of D under $\zeta = \log z$ is thus the half-strip $0 < \xi < \infty$, $0 < \eta < \pi$, as shown in Figure 1.28.

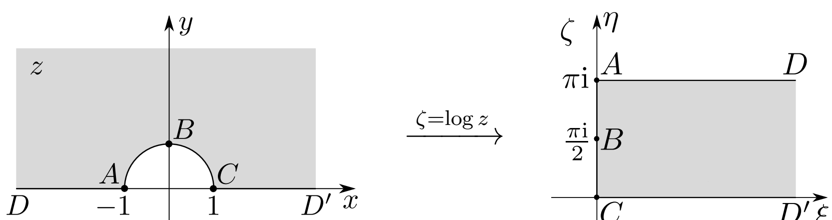

Figure 1.28: The logarithmic map transforms a half-plane with a semi-circle removed to a half-strip.

Trigonometric maps

Like the exponential, the sine and cosine functions are periodic and have essential singularities. Unlike the exponential, they have derivatives that vanish. Hence these functions can combine the properties of opening out (doubling) angles on the boundary of D with the useful strip-mapping properties of the exponential. Their inverses have similar uses to the logarithm.

Example. Investigate the effect of the mapping $\zeta = \cos z$ on the half-strip $0 < y < \infty$, $0 < x < \pi$. Calculate the image of the lines $y = \text{constant}$.

Solution. The critical points where the derivative vamishes are at $z = 0, \pi$. Hence the internal angles of ∂D are doubled there. Note that

$$\cos z = \cos x \cosh y - \mathrm{i} \sin x \sinh y. \tag{1.61}$$

When z is real, $\cos z$ runs from 1 at $z = 0$ to -1 at $z = \pi$. Thus the line segment $z \in (0, 1)$ maps to the interval $-1 < \xi < 1$ of the ξ axis, and the image of D is below this line. The line $x = 0$, $y > 0$ maps to $\zeta = \cos \mathrm{i}y = \cosh y$, which traces the ξ axis from 1 to $+\infty$ The line $x = \pi$, $y > 0$ maps to the segment $-\infty < \xi = -\cosh y < -1$ of the ξ axis. Thus $\zeta = \cos z$ maps the half-strip to the lower half-plane, with the two critical points mapping to $\zeta = \pm 1$.

A line $y =$ constant, maps to the curve given parametrically by

$$\xi = \cos x \cosh y, \qquad \eta = -\sin x \sinh y, \qquad x \in (0, \pi), \tag{1.62}$$

namely the lower half of the ellipse

$$\frac{\xi^2}{\cosh^2 y} + \frac{\eta^2}{\sinh^2 y} = 1. \tag{1.63}$$

Similarly, the lines $x =$ constant map to the hyperbolae

$$\frac{\xi^2}{\cos^2 x} - \frac{\eta^2}{\sin^2 x} = 1, \tag{1.64}$$

which are orthogonal to the ellipses (1.63), as shown in Figure 1.29.

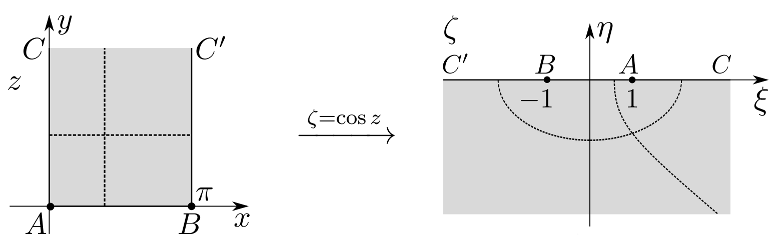

Figure 1.29: The map $\zeta = \cos z$ maps lines $x =$ constant and $y =$ constant are to orthogonal families of hyperbolae and ellipses.

Example: the tangent function. Show that the mapping $\zeta = \tan z$ maps the strip $-\pi/4 < x < \pi/4$, $-\infty < y < \infty$ to the unit disc.

Solution. It would be a mistake to think of $\tan z$ as the ratio $\sin z / \cos z$: the product of two conformal mappings has no natural geometric meaning. However, by writing

$$\tan z = \frac{\sin z}{\cos z} = -\mathrm{i} \frac{\mathrm{e}^{2\mathrm{i}z} - 1}{\mathrm{e}^{2\mathrm{i}z} + 1}, \tag{1.65}$$

we can view the tangent map as a composition between exponential and Möbius transformations, via the sequence

$$\zeta_1 = \mathrm{e}^{2\mathrm{i}z}, \qquad \zeta = -\mathrm{i} \frac{\zeta_1 - 1}{\zeta_1 + 1}, \tag{1.66}$$

as illustrated in Figure 1.30.

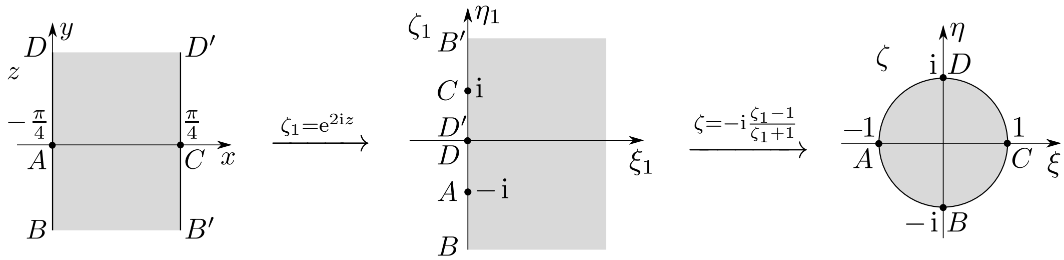

Figure 1.30: The tangent mapping viewed as the composition of an exponential with a Möbius transformation.

The Joukowski map

Our final example is the map called the *Joukowski map*,

$$\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{1.67}$$

The Joukowski map has derivative equal to zero at two critical points $z = \pm 1$. The image of the unit circle $|z| = 1$ is the slit $-1 < \xi < 1$ of the real axis, as when $z = e^{i\theta}$, $\zeta = \cos\theta$. The exterior of the unit circle is mapped to the whole plane exterior to this slit (which is a branch cut for the inverse mapping). Similarly the interior of the unit circle is mapped to the whole plane exterior to the same slit. A circle $|z| = \rho > 1$ is mapped to an ellipse, and a ray $\arg z =$ constant to a member of the orthogonal family of hyperbolae, as shown in Figure 1.31.

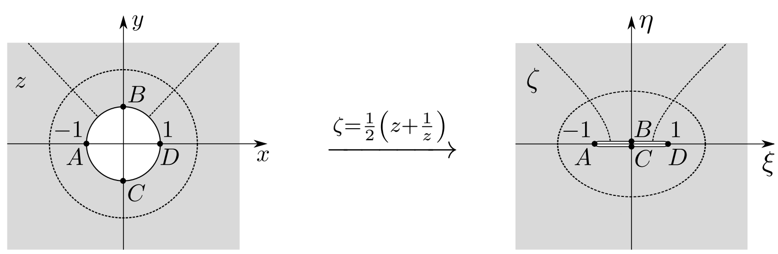

Figure 1.31: The effects of the Joukowski mapping on the region outside the unit circle.