4 Plemelj formulae and applications

4.1 Introduction

The problem of determining a holomorphic function $w(z)$ in terms of its values on a curve Γ is equivalent to solving a Cauchy problem for Laplace's equation and therefore *ill-posed*: the solution may not exist or may not be unique or it may not depend continuously on the boundary values.

Example. If $w(z)$ is holomorphic in $y > 0$ and

$$w(x) = \frac{\delta^2 \epsilon}{\delta^2 + x^2} \quad \text{for } y = 0, \ -\infty < x < \infty, \tag{4.1}$$

then

$$w(z) = \frac{\delta^2 \epsilon}{\delta^2 + z^2}. \tag{4.2}$$

Thus $|w| \le \epsilon$ on $y = 0$, and $w \to \infty$ as $z \to \mathrm{i}\delta$. Since ϵ and δ may be arbitrarily small, we see that, however small w is on $y = 0$, it may become arbitrarily large an arbitrarily small distance from $y = 0$.

This example illustrates that trying to specify $w(z)$ on a given curve is ill posed. However, well-posed problems may be formulated in which, for example, $\operatorname{Re} w$ or $\operatorname{Im} w$ are specified on Γ or the jump in w across Γ is prescribed. We will show how a wide class of such problems may be tackled using the so-called *Plemelj formulae.*

4.2 Plemelj formulae

Recall that if w is holomorphic inside and on the closed contour Γ and z is a point inside Γ, then Cauchy's integral formula states that

$$w(z) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{w(\zeta)\,\mathrm{d}\zeta}{\zeta - z}. \tag{4.3}$$

This relates the values of w inside the contour to the values of w on the contour.

Let us consider more generally the *Cauchy integral*

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}, \tag{4.4}$$

where f is a given function on the contour Γ, which may now be closed or open. If Γ is open, it is convenient in the subsequent analysis to adopt the convention that it does not contain its endpoints, a, $b \in \mathbb{C}$ say. Thus, an open contour may be parametrized by

$$\Gamma = \{\gamma(t) \in \mathbb{C} : t_0 < t < t_1\}, \tag{4.5}$$

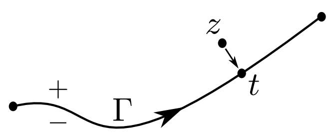
where $a = \gamma(t_0) \neq \gamma(t_1) = b$ and $t_0 < t_1$ are real constants. We then define

$$\overline{\Gamma} = \{\gamma(t) \in \mathbb{C} : t_0 \leq t \leq t_1\} \tag{4.6}$$

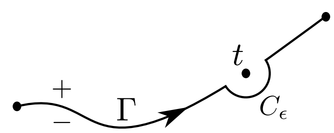
to be the (topological) *closure* of Γ, i.e. $\overline{\Gamma}$ is the union of Γ and its endpoints. (If Γ is a closed contour, then $\overline{\Gamma} = \Gamma$ because Γ is (topologically) closed.)

If f is sufficiently smooth (e.g. continuous) on $\overline{\Gamma}$, then the function $w(z)$ defined by the Cauchy integral (4.4) is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$ (its derivatives may be found by differentiating under the integral sign). Now we pose the question: what is the limiting value of $w(z)$ as z approaches Γ? It turns out that the answer depends on which side of Γ is approached by z.

Suppose $t \in \Gamma$ is any point at which Γ is smooth and that f is holomorphic in a neighbourhood of t and continuous on Γ. Let us label the left-hand side of Γ (as Γ is traversed in the direction of integration) as "+", and the right-hand side as "−". Let z approach $t \in \Gamma$ from the positive side as illustrated in Figure 4.1(a). We deform Γ near t by replacing



(a) Original contour



(b) Deformed contour

Figure 4.1: Deformed integration contour for $w_+(z)$.

$\gamma_\epsilon = \Gamma \cap D(t;\epsilon) \subset \Gamma$ with a small semi-circle C_ϵ as illustrated in Figure 4.1(b), where ϵ is sufficiently small that f is holomorphic in the disc $D(t;2\epsilon) = \{z : |z-t| < 2\epsilon\}$ say. By the deformation theorem,

$$w_+(t) = \lim_{z\to t} \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma\setminus\gamma_\epsilon} + \int_{C_\epsilon} \right) \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma\setminus\gamma_\epsilon} + \int_{C_\epsilon} \right) \frac{f(\zeta)}{\zeta - t}\,\mathrm{d}\zeta. \tag{4.7}$$

As $\epsilon \to 0$, the semi-circle gives a residue contribution

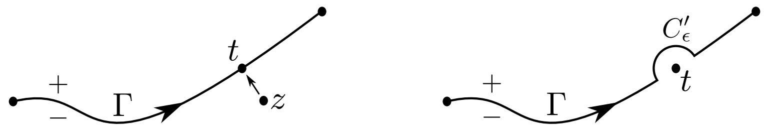
$$\frac{1}{2} \times 2\pi \mathrm{i} \times \frac{f(t)}{2\pi \mathrm{i}} = \frac{1}{2} f(t),$$

where the factor of $1/2$ arises because we are only integrating over a semi-circle. Hence,

$$w_+(t) = \lim_{\epsilon\to 0} \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma\setminus\gamma_\epsilon} + \int_{C_\epsilon} \right) \frac{f(\zeta)}{\zeta - t}\,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} ⨍_{\Gamma} \frac{f(\zeta)}{\zeta - t}\,\mathrm{d}\zeta + \frac{1}{2} f(t), \tag{4.8}$$

where we define the *Principal Value integral* as

$$⨍_{\Gamma} \frac{f(\zeta)}{\zeta - t}\,\mathrm{d}\zeta = \lim_{\epsilon\to 0} \int_{\Gamma\setminus\gamma_\epsilon} \frac{f(\zeta)}{\zeta - t}\,\mathrm{d}\zeta. \tag{4.9}$$



(a) Original contour (b) Deformed contour

Figure 4.2: Deformed integration contour for $w_-(z)$.

This limit always exists because the log singularities from the endpoints cancel as $\epsilon \to 0$ when f is continuous on Γ.

If we let $z \to t \in \Gamma$ from the minus side as illustrated in Figure 4.2(a), then we must deform Γ near $\zeta = t$ by replacing $\gamma_\epsilon \subset \Gamma$ with a small semi-circle C'_ϵ as illustrated in Figure 4.2(b). Again by the deformation theorem

$$w_-(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma \setminus \gamma_\epsilon} + \int_{C'_\epsilon} \right) \frac{f(\zeta)}{\zeta - t} \, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)}{\zeta - t} \, \mathrm{d}\zeta - \frac{1}{2} f(t). \tag{4.10}$$

In this case we are integrating in the opposite direction around the semi-circle, so that the residue contribution is $-f(t)/2$.

Equations (4.8) and (4.10) are known as the **Plemelj formulae**. In deriving them, we have assumed that Γ is a smooth contour and that f is continuous on $\overline{\Gamma}$. These conditions may be relaxed (see e.g. Ablowitz & Fokas), but we will persist with these assumptions henceforth. It follows that $w(z)$ is holomorphic and that $w(z) = O(1/z)$ as $z \to \infty$.

The contour deformation approach shown in Figures 4.1 and 4.2 clearly does not work if $t = t_e$ ($= a$ or b) is an end-point of Γ. The local behaviour as $z \to t_e$ depends on the local behaviour of $f(\zeta)$. The following results may be derived using perturbation methods or quoted from Ablowitz & Fokas.

As $z \to t_e$ with $z \in \mathbb{C} \setminus \overline{\Gamma}$:

if $f(\zeta) \to 0$ as $\zeta \to t_e$, then $w(z) = O(1)$; (4.11a)

if $f(\zeta) = O(1)$ as $\zeta \to t_e$, then $w(z) = O\big(\log(z - t_e)\big)$; (4.11b)

if $f(\zeta) = O\big((\zeta - t_e)^{-\alpha}\big)$ as $\zeta \to t_e$, with $\alpha \in (0, 1)$, then $w(z) = O\big((z - t_e)^{-\alpha}\big)$. (4.11c)

4.3 Solving problems with the Plemelj formulae

Problem 1

Find a function $w(z)$ holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$ such that the limiting values of $w(z)$ as $z \to t \in \Gamma$ from either side satisfy

$$w_+(t) - w_-(t) = G(t), \tag{4.12}$$

where G is continuous on $\overline{\Gamma}$.

Solution. We seek a solution for w as a Cauchy integral

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta) \, \mathrm{d}\zeta}{\zeta - z}, \tag{4.13}$$

where our aim is to use the jump condition (4.12) to determine the density function f. By subtracting the Plemelj formulae (4.10) and (4.8) we find that

$$w_+(t) - w_-(t) = f(t) \tag{4.14}$$

on Γ. Hence, we read off $f = G$, and a solution is given by

$$w(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{G(\zeta)\,\mathrm{d}\zeta}{\zeta - z}. \tag{4.15}$$

This shows that the Plemelj formulae allow us easily to find *a* solution $w(z)$ that is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$ and satisfies the jump condition (4.12). However, the solution (4.15) is not unique. The homogeneous problem with $G = 0$ consists of finding a function that is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$ and continuous across Γ, which is satisfied by *any* function $w(z) = h(z)$ that is holomorphic on $\mathbb{C}\setminus\{a, b\}$. Morera's Theorem may be used to prove that *all* solutions of the homogeneous problem must be of this form. Therefore the general solution of Problem 1 is

$$w(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{G(\zeta)\,\mathrm{d}\zeta}{\zeta - z} + h(z), \tag{4.16}$$

where $h(z)$ is an arbitrary function of z that is holomorphic on $\mathbb{C}\setminus\{a, b\}$.

To pin down h, it is necessary to prescribe the behaviour of w at a, b and ∞. For example, suppose we impose the additional conditions:

(I) w is finite or has a logarithmic singularity at each of the endpoints of Γ;

(II) there exists $n \in \mathbb{N}$ such that $w(z) = O(z^n)$ as $|z| \to \infty$.

Then, (I), the quotable results (4.11) and Laurent's Theorem imply that h can only have removable singularities at a and b, so that h is in fact entire. Hence, by (II) and the corollary to Liouville's theorem, $h(z) = p_n(z)$, an arbitrary polynomial of degree n.

Problem 2

Consider the particular case where Γ is a line segment on the real axis: $\Gamma = \{x : 0 < x < c\}$ for some $c > 0$. Suppose we are given $\operatorname{Im} w_\pm(x) = g_\pm(x)$ on Γ, with w holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$. Find w when (1) $g_+(x) = -g_-(x) = g(x)$ and (2) $g_+(x) = g_-(x) = g(x)$, where $g(x)$ is continuous on $\overline{\Gamma}$.

Remark. If $w(z) = u(x, y) + \mathrm{i}v(x, y)$, then this problem is equivalent to the problem of finding v such that $\nabla^2 v = 0$ away from $\overline{\Gamma}$, and $v_\pm(x) = g_\pm(x)$ on Γ.

Solution. Seek a solution for w as a Cauchy integral of the form

$$w(z) = \frac{1}{2\pi\mathrm{i}} \int_0^c \frac{f(\xi)\,\mathrm{d}\xi}{\xi - z}, \tag{4.17}$$

which is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$, assuming f is sufficiently regular. The Plemelj formulae (4.8)–(4.10) become

$$w_\pm(x) = u_\pm(x) + \mathrm{i}g_\pm(x) = \pm\frac{1}{2} f(x) - \mathrm{i}F(x) \quad \text{on } \Gamma, \tag{4.18}$$

where we define

$$F(x) = \frac{1}{2\pi} ⨍_0^c \frac{f(\xi)}{\xi - x} \mathrm{d}\xi. \tag{4.19}$$

Note that $F(x)$ is real on Γ if and only if $f(x)$ is real on Γ (because ξ, x are real on Γ).

Problem 2.1: If $g_+(x) = -g_-(x) = g(x)$, then (4.18) implies that

$$w_+(x) + w_-(x) = u_+(x) + u_-(x) = -2\mathrm{i}F(x) \qquad \text{on } \Gamma, \tag{4.20a}$$
$$w_+(x) - w_-(x) = u_+(x) - u_-(x) + 2\mathrm{i}g(x) = f(x) \qquad \text{on } \Gamma. \tag{4.20b}$$

By (4.20a), F must be pure imaginary, and hence f must be pure imaginary on Γ. Thus, by (4.20b), we have $u_+(x) - u_-(x) = 0$ and $f(x) = 2\mathrm{i}g(x)$ on Γ. It follows that a solution for w is given by

$$w(z) = \frac{1}{\pi} \int_0^c \frac{g(\xi)\,\mathrm{d}\xi}{\xi - z} + h(z), \tag{4.21}$$

where $h(z)$ is an arbitrary function of z that is holomorphic on $\mathbb{C} \setminus \{0,\, c\}$ and real on Γ (thus a solution of the homogeneous problem in which $g = 0$).

Problem 2.2: If $g_+(x) = g_-(x) = g(x)$, then (4.18) becomes

$$w_+(x) + w_-(x) = u_+(x) + u_-(x) + 2\mathrm{i}g(x) = -2\mathrm{i}F(x) \qquad \text{on } \Gamma, \tag{4.22a}$$
$$w_+(x) - w_+(x) = u_+(x) - u_-(x) = f(x) \qquad \text{on } \Gamma. \tag{4.22b}$$

By (4.22b), f must be real, and hence F must likewise be real, on Γ; thus, by (4.22a), we have $u_+(x) + u_-(x) = 0$ and $F(x) = -g(x)$ on Γ. It follows that

$$w(z) = \frac{1}{2\pi\mathrm{i}} \int_0^c \frac{f(\xi)\,\mathrm{d}\xi}{\xi - z} \tag{4.23}$$

is a solution provided f satisfies the *Cauchy singular integral equation*

$$\frac{1}{\pi} ⨍_0^c \frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} = -2g(x) \qquad (0 < x < c), \tag{4.24}$$

which we need to invert to find f.

Remark: In Problem 2.1 the data gives $w_+ - w_-$ and hence f directly. In Problem 2.2 the data gives $w_+ + w_-$ leading to a Cauchy singular integral equation for f.

Solution. Suppose we can find an auxillary function $\tilde{w}(z)$ such that:

- $\tilde{w}(z)$ is holomorphic and non-zero on $\mathbb{C} \setminus \overline{\Gamma}$; (4.25a)
- $\tilde{w}(z)$ satisfies $\tilde{w}_+(x) = -\tilde{w}_-(x) \neq 0$ on Γ, (4.25b)

i.e. $\tilde{w}$ is a solution of the homogeneous problem (in which $g = 0$) that is non-zero on $\mathbb{C} \setminus \{a, b\}$.

Now we define

$$W(z) = \frac{w(z)}{\tilde{w}(z)}, \tag{4.26}$$

so that

$$
\begin{aligned}
W_+(x) - W_-(x) &= \frac{w_+(x)}{\tilde{w}_+(x)} - \frac{w_-(x)}{\tilde{w}_-(x)} \\
&= \frac{w_+(x)}{\tilde{w}_+(x)} - \frac{w_-(x)}{-\tilde{w}_+(x)} \\
&= \frac{w_+(x) + w_-(x)}{\tilde{w}_+(x)} \\
&= \frac{2\mathrm{i}g(x)}{\tilde{w}_+(x)} \quad \text{on } \Gamma. \qquad (4.27)
\end{aligned}
$$

If $\tilde{w}_+$ is known, then $W_+ - W_-$ is known (because g is known). Therefore we have turned Problem 2.2 (in which $w_+ + w_-$ is given) into a version of Problem 1 (in which $W_+ - W_-$ is given). By Problem 1, equation (4.15), a solution for W is given by

$$
W(z) = \frac{1}{2\pi\mathrm{i}} \int_0^c \frac{\tilde{f}(\xi)\,\mathrm{d}\xi}{\xi - z} + \tilde{H}(z), \qquad (4.28)
$$

where

$$
\tilde{f}(x) = \frac{2\mathrm{i}g(x)}{\tilde{w}_+(x)} \quad \text{on } \Gamma, \qquad (4.29)
$$

and $\tilde{H}(z)$ is an arbitrary function holomorphic on $\mathbb{C} \setminus \{0, c\}$. Thus the solution of Problem 2.2 takes the form

$$
w(z) = \tilde{w}(z) \left(\frac{1}{\pi} \int_0^c \frac{g(\xi)\,\mathrm{d}\xi}{\tilde{w}_+(\xi)(\xi - z)} + \tilde{H}(z) \right). \qquad (4.30)
$$

With W given by (4.28), the Plemelj formulae give

$$
W_\pm(x) = \pm\frac{1}{2}\tilde{f}(x) + \frac{1}{2\pi\mathrm{i}} \fint_0^c \frac{\tilde{f}(\xi)\,\mathrm{d}\xi}{\xi - x} + \tilde{H}(x) \qquad (0 < x < c), \qquad (4.31)
$$

so that

$$
\tilde{f}(x) = W_+(x) - W_-(x) = \frac{2\mathrm{i}g(x)}{\tilde{w}_+(x)} \quad \text{on } \Gamma, \qquad (4.32)
$$

as required. Moreover,

$$
\begin{aligned}
\frac{1}{\pi\mathrm{i}} \fint_0^c \frac{\tilde{f}(\xi)\,\mathrm{d}\xi}{\xi - x} + 2\tilde{H}(x) &= W_+(x) + W_-(x) \\
&= \frac{w_+(x)}{\tilde{w}_+(x)} + \frac{w_-(x)}{\tilde{w}_-(x)} \\
&= \frac{w_+(x) - w_-(x)}{\tilde{w}_+(x)} \\
&= \frac{f(x)}{\tilde{w}_+(x)} \quad \text{on } \Gamma, \qquad (4.33)
\end{aligned}
$$

and, with $\tilde{f}$ given by (4.29), we deduce that

$$
f(x) = \tilde{w}_+(x)\big(W_+(x) + W_-(x)\big) = 2\tilde{w}_+(x) \left(\frac{1}{\pi} \fint_0^c \frac{g(\xi)\,\mathrm{d}\xi}{\tilde{w}_+(\xi)(\xi - x)} + \tilde{H}(x) \right) \qquad (4.34)
$$

satisfies the Cauchy singular integral equation (4.24).

Finding $\tilde{w}$

We have shown that the decomposition (4.26) allows us to transform Problem 2.2 into a version of Problem 1, and then solve it using the Plemelj formulae. As a bonus, (4.34) gives the solution $f(x)$ of the singular integral equation (4.24). It just remains to find an auxillary function $\tilde{w}(z)$ satisfying the properties (4.25), where $\Gamma = \{x + \mathrm{i}y : 0 < x < c,\, y = 0\}$ and $\overline{\Gamma} = \{x + \mathrm{i}y : 0 \le x \le c,\, y = 0\}$. We need to find a function whose value as Γ is approached from above is minus that as Γ is approached from below, as shown schematically in Figure 4.3.

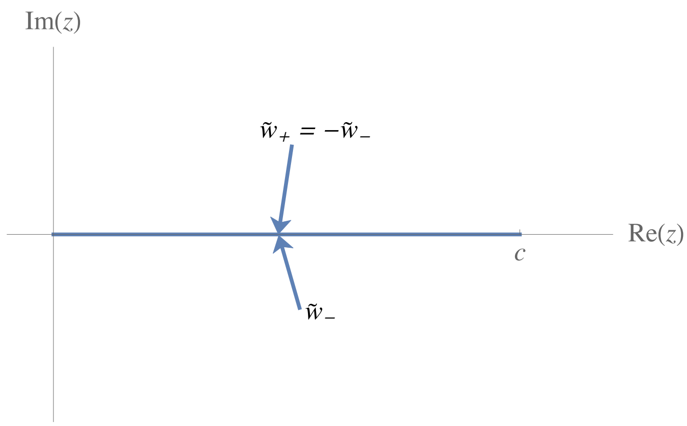


Figure 4.3: The jump conditions satisfied by the auxiliary function across Γ.

Example 1. When $c = \infty$, we can use $\tilde{w}(z) = z^{1/2}$, provided we take the branch cut along the positive real axis, i.e. $z^{1/2} = r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}$ for $z = r\mathrm{e}^{\mathrm{i}\theta}$, with $r > 0$ and $0 < \theta \le 2\pi$. Then we will have $\tilde{w}_{\pm}(x) = \pm x^{1/2} \ne 0$ for $x > 0$, as required. We can obtain another valid solution by multiplying $\tilde{w}(z)$ by any function of z that is holomorphic and non-zero on $\mathbb{C} \setminus \{0\}$.

Example 2. When $0 < c < \infty$, we can use $\tilde{w}(z) = z^{1/2}(c - z)^{1/2}$, where we take the branch cut along Γ and then $\tilde{w}_{\pm}(x) = \pm x^{1/2}(c - x)^{1/2} \ne 0$ for $0 < x < c$. In this case, we can obtain another valid solution by multiplying $\tilde{w}(z)$ by any function of z that is holomorphic and non-zero on $\mathbb{C} \setminus \{0, c\}$.

In the above two examples, the auxiliary function $\tilde{w}(z)$ could plausibly have been found by inspection. However, we might wonder whether the functions so obtained are unique, and also how one could find $\tilde{w}$ more generally. We have $\tilde{w}_+/\tilde{w}_- = -1$ on Γ, so

$$\log \tilde{w}_+ - \log \tilde{w}_- = \log(-1) = (2m + 1)\pi\mathrm{i} \quad \text{on } \Gamma, \tag{4.35}$$

where $m \in \mathbb{Z}$, corresponding to the infinite number of branches of the logarithm. Equation (4.35) is a version of Problem 1, and we read off from equations (4.12) and (4.16) the solution

$$\begin{aligned} \log \tilde{w}(z) &= \frac{1}{2\pi\mathrm{i}} \int_0^c \frac{(2m + 1)\pi\mathrm{i}}{\xi - z}\,\mathrm{d}\xi + \tilde{h}(z) \\ &= \left(m + \frac{1}{2}\right)\big[\log(c - z) - \log z\big] + \tilde{h}(z), \end{aligned} \tag{4.36}$$

where $\tilde{h}(z)$ is an arbitrary function holomorphic on $\mathbb{C} \setminus \{0, c\}$. Therefore the general form for $\tilde{w}(z)$ is

$$\tilde{w}(z) = h^*(z) \left(\frac{c - z}{z} \right)^{m+1/2}, \tag{4.37}$$

where $h^*(z) = \mathrm{e}^{\tilde{h}(z)}$ is again an arbitrary function of z holomorphic and nonzero on $\mathbb{C} \setminus \{0, c\}$. The general solution (4.37) includes the particular form for $\tilde{w}$ found in Example 2 above, with $m = 0$ and $h^*(z) = z$.

Evidently the solution of Problem 2.2 is far from unique. There is a lot of freedom in the general form (4.37) for $\tilde{w}$, and also the arbitrary function $\tilde{H}(z)$ in (4.30) must be determined. We will now work through two concrete examples to show how a unique solution may be selected by prescribing the allowed behaviour of $w(z)$ at $z = 0$, $z = c$ and as $z \to \infty$.

4.4 Example: Fracture in solid mechanics

A famous problem in elasticity is to calculate the displacement field $\big(0, 0, \Phi(x, y)\big)$ in antiplane strain around a crack at $y = 0$, $0 < x < c$, as illustrated in Figure 4.4(a). The displacement Φ is such that:

- $\nabla^2 \Phi = 0$ except on the crack;
- $\lim_{y \downarrow \uparrow 0} \partial\Phi/\partial y = 0$ for $0 < x < c$ (zero traction on the crack surface);
- $|\nabla\Phi|$ has an inverse square-root singularity at $(0, 0)$ and at $(c, 0)$ (so that the displacement Φ is finite at the crack tips);
- $\partial\Phi/\partial y = T + O\left(r^{-2}\right)$ as $r^2 = x^2 + y^2 \to \infty$ (uniform shearing at large distances).

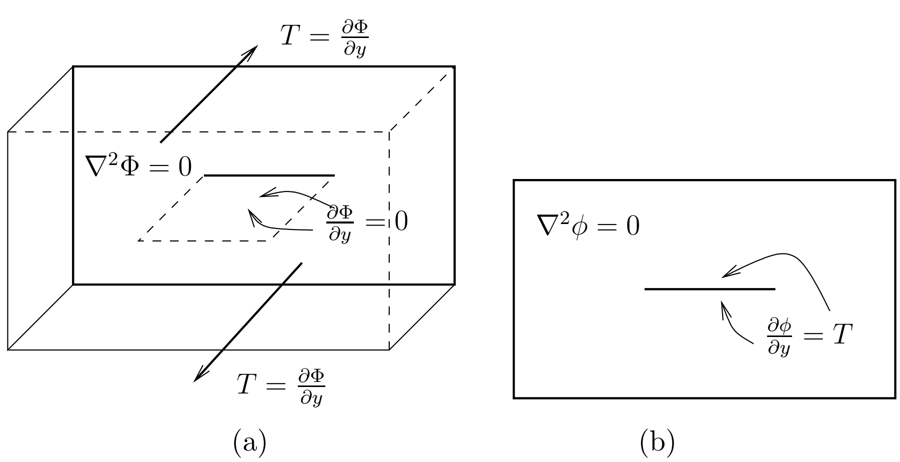


Figure 4.4: (a) Antiplane strain around a crack. (b) The two-dimensional problem for $\phi(x, y)$.

Setting $\Phi = Ty - \phi(x, y)$ and $\phi_y = \operatorname{Im} w(z)$, we find that the corresponding properties of w are:

- $w(z)$ is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$;

- $\operatorname{Im} w_\pm(x) = T$ on $\Gamma = \{x + \mathrm{i}y : 0 < x < c,\ y = 0\}$;
- $w(z) = O\left(z^{-1/2}\right)$ as $z \to 0$ and $w(z) = O\left((z-c)^{-1/2}\right)$ as $z \to c$;
- $w(z) = O(z^{-2})$ as $z \to \infty$.

This is equivalent to Problem 2.2, with $g(x) = T =$ constant, so a solution is given by equation (4.30), namely

$$w(z) = \tilde{w}(z)\left(\frac{1}{\pi}\int_0^c \frac{g(\xi)\,\mathrm{d}\xi}{\tilde{w}_+(\xi)(\xi - z)} + \tilde{H}(z)\right), \tag{4.38}$$

where $\tilde{H}(z)$ is an arbitrary function of z holomorphic on $\mathbb{C}\backslash\{0, c\}$. We now make a specific choice for $\tilde{w}$, namely

$$\tilde{w}(z) = z^{-1/2}(c - z)^{-1/2}, \tag{4.39}$$

with the branch cut along Γ, so that $\tilde{w}_\pm(x) = \pm x^{-1/2}(c - x)^{-1/2}$ for $0 < x < c$, and equation (4.38) becomes

$$w(z) = \frac{1}{z^{1/2}(c - z)^{1/2}}\left(\frac{1}{\pi}\int_0^c \frac{\xi^{1/2}(c - \xi)^{1/2}g(\xi)}{(\xi - z)}\,\mathrm{d}\xi + \tilde{H}(z)\right). \tag{4.40}$$

Now we will use the prescribed properties of $w(z)$ to argue that $\tilde{H}(z)$ must in fact be zero.

- At the endpoints $z = 0$ and $z = c$ of Γ, the integral in (4.40) is finite (because of the choice we made for $\tilde{w}(z)$).
- Since $\tilde{H}(z)$ is holomorphic on $\mathbb{C}\backslash\{0,\, c\}$, it can only have *isolated* singularities at the end points.
- Since $w = O\left(z^{-1/2}\right)$ as $z \to 0$ and $w = O\left((c - z)^{-1/2}\right)$ as $z \to c$, it follows that $\tilde{H}(z)$ can only have *removable* singularities at $z = 0$ and $z = c$, and therefore $\tilde{H}(z)$ is *entire*.
- Finally, $w = O\left(z^{-2}\right)$ as $z \to \infty$ if and only if $\tilde{H}(z) = O\left(z^{-1}\right)$ as $z \to \infty$, and therefore $\tilde{H}(z) \equiv 0$ by Liouville's theorem.

Hence, the unique solution for $w(z)$ is given by

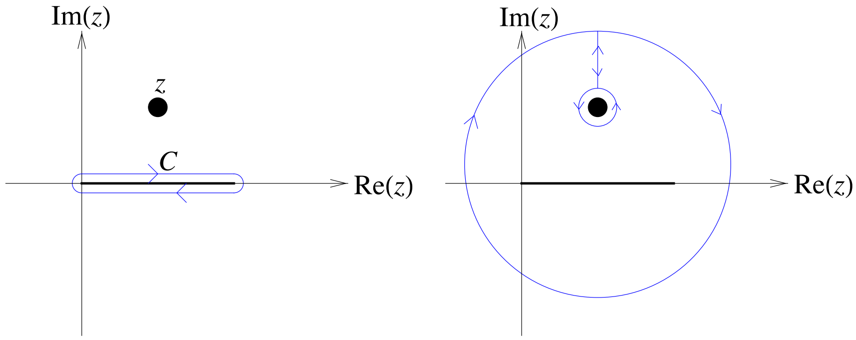
$$w(z) = \frac{T}{\pi z^{1/2}(c - z)^{1/2}}\int_0^c \frac{\xi^{1/2}(c - \xi)^{1/2}\,\mathrm{d}\xi}{(\xi - z)}. \tag{4.41}$$

The integral in equation (4.41) can be evaluated explicitly as follows. First note that

$$\int_0^c \frac{\xi^{1/2}(c - \xi)^{1/2}\,\mathrm{d}\xi}{(\xi - z)} = \frac{1}{2}\oint_C \frac{\zeta^{1/2}(c - \zeta)^{1/2}\,\mathrm{d}\zeta}{(\zeta - z)}, \tag{4.42}$$

where C is a small clockwise contour that encloses Γ, as shown in Figure 4.5(a). Now deform the contour C to infinity, as shown in Figure 4.5(b). There is a residue contribution from the pole at $\zeta = z$ of $\pi\mathrm{i}z^{1/2}(c - z)^{1/2}$. To evaluate the contribution from a large circle at infinity expand the integrand as

$$\frac{\zeta^{1/2}(c - \zeta)^{1/2}}{(\zeta - z)} \sim -\mathrm{i}\left(1 - \frac{c}{\zeta}\right)^{1/2}\left(1 - \frac{z}{\zeta}\right)^{-1} \sim -\mathrm{i}\left(1 + \frac{2z - c}{2\zeta} + \cdots\right) \tag{4.43}$$



(a) (b)

Figure 4.5: Integration contours for the integral (4.41).

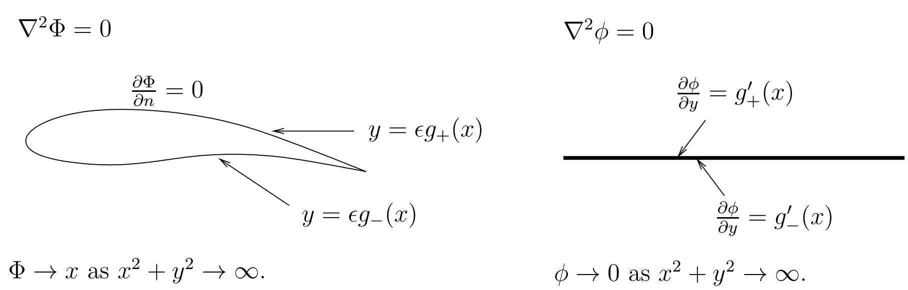
which integrates to $-\pi(z - c/2)$. Thus the explicit solution for $w(z)$ is

$$w = \frac{T}{\pi z^{1/2}(c-z)^{1/2}} \left(\pi \mathrm{i} z^{1/2}(c-z)^{1/2} - \pi\left(z - \frac{c}{2}\right)\right) = T\mathrm{i} - \frac{T(z - c/2)}{z^{1/2}(c-z)^{1/2}}. \qquad (4.44)$$

We can easily verify that the solution (4.44) for $w(z)$ has all of the required properties. In principle we would have obtained exactly the same solution if we made a different choice of the auxiliary function $\tilde{w}(z)$: it would just have made the job of determining $\tilde{H}(z)$ slightly more difficult. In general, a judicious choice of $\tilde{w}(z)$ will make the whole solution procedure as straightforward as possible.

4.5 Example: Aerodynamics of a thin aerofoil

Here the physical model is the flow of a uniform stream of ideal fluid past a thin aerofoil with a sharp trailing edge and a small angle of attack, as illustrated in Figure 4.6(a). We denote



(a) (b)

Figure 4.6: Flow past a thin aerofoil. (a) The problem for the velocity potential $\Phi(x, y)$. (b) The linearised problem for the disturbance potential $\phi(x, y)$.

the boundary of the aerofoil by $y = \epsilon g_\pm(x)$ for $0 < x < c$, where $g_-(x) \le g_+(x)$ and $\epsilon \ll 1$. If $\Phi(x, y)$ is the velocity potential, then:

- $\nabla^2\Phi = 0$ in the fluid surrounding the aerofoil;

- the no-flux boundary condition states that $\partial\Phi/\partial n = 0$ on the boundary of the aerofoil;
- there is an *inverse square root* singularity in the velocity at the leading edge, so that $|\boldsymbol{\nabla}\Phi| = O\left(r^{-1/2}\right)$ as $r = \sqrt{x^2+y^2} \to 0$;
- the *Kutta condition* states that the velocity $\boldsymbol{\nabla}\Phi$ must be finite at the sharp trailing edge;
- the velocity is uniform at infinity, so that $\boldsymbol{\nabla}\Phi \sim (1,0) + O\left(r^{-1}\right)$ as $r \to \infty$.

In the limit of a thin aerofoil, $\epsilon \to 0$ and we can expand about the uniform flow, setting $\Phi(x,y) \sim x + \epsilon\phi(x,y)$. Since the outward normal to the upper surface of the aerofoil is proportional to $(-\epsilon g_+', 1)$, the no-flux boundary condition on the upper surface implies

$$\begin{aligned} 0 &= (-\epsilon g_+', 1)\cdot\nabla\Phi \quad \text{on } y = \epsilon g_+(x) \\ &= (-\epsilon g_+', 1)\cdot\big(1+\epsilon\phi_x(x,\epsilon g_+), \phi_y(x,\epsilon g_+)\big) \\ &\sim -\epsilon g_+' + \epsilon\phi_y(x,0) + O(\epsilon^2) \end{aligned} \tag{4.45}$$

as $\epsilon \to 0$. A similar expansion holds for the no-flux boundary condition on the lower surface. Thus the boundary conditions which were originally imposed on the surface of the aerofoil may be linearised down onto the x-axis when ϵ is small.

The leading-order problem for the disturbance potential $\phi(x,y)$ is:

- $\nabla^2\phi = 0$ except on the line segment $\{(x,y) : 0 \le x \le c,\ y = 0\}$;
- $\dfrac{\partial\phi}{\partial y} = g_\pm'(x) \quad$ on $0 < x < c$, $y = 0_\pm$;
- $|\boldsymbol{\nabla}\phi| = O\left(r^{-1/2}\right)$ as $r \to 0$;
- $\boldsymbol{\nabla}\phi$ is finite as $(x,y) \to (c,0)$;
- $|\boldsymbol{\nabla}\phi| = O\left(r^{-1}\right)$ as $r \to \infty$.

We translate this into a Plemelj type problem by defining

$$w(z) = -\big(\phi_x(x,y) - \mathrm{i}\phi_y(x,y)\big) \tag{4.46}$$

(the unconventional minus sign is taken for convenience). Then $w(z)$ has the properties

- $w(z)$ is holomorphic on $\mathbb{C}\setminus\overline{\Gamma}$;
- $\operatorname{Im} w_\pm(x) = g_\pm'(x)$ on $\Gamma = \{x + \mathrm{i}y : 0 < x < c,\ y = 0\}$;
- $w(z) = O(z^{-1/2})$ as $z \to 0$ and $w(z) = O(1)$ as $z \to c$;
- $w(z) = O\left(z^{-1}\right)$ as $z \to \infty$.

	fracture	aerofoil
$z \to 0$	$w(z) = O\left(z^{-1/2}\right)$	$w(z) = O\left(z^{-1/2}\right)$
$z \to c$	$w(z) = O\left((z-c)^{-1/2}\right)$	$w(z) = O\left(1\right)$
$z \to \infty$	$w(z) = O\left(z^{-2}\right)$	$w(z) = O\left(z^{-1}\right)$

Table 1: Comparison between the prescribed behaviours of $w(z)$ in the fracture and aerofoil problems.

Figure 4.7: Schematic of a symmetric aerofoil (left); a zero-thickness aerofoil (right).

Remark: In Table 1 we summarise the conditions specified for $w(z)$ at $z = 0$, $z = c$ and as $z \to \infty$ in the fracture and aerofoil problems. Compared with the fracture problem, we have now *strengthened* the condition at $z = c$ but *weakened* the condition at infinity.

For a *symmetric* aerofoil, $g_+(x) = -g_-(x)$, so that $g_+'(x) = -g_-'(x)$ and we must solve an easy problem as in Problem 2.1. A zero-thickness aerofoil has $g_+(x) = g_-(x)$, as shown in Figure 4.7, so that $g_+'(x) = g_-'(x)$ and we must solve a harder problem as in Problem 2.2.

In the latter case, we let $g_+'(x) = g_-'(x) = g(x)$ and again choose $\tilde{w}(z) = z^{-1/2}(c-z)^{-1/2}$, so that we can use the same solution (4.40) as for the crack problem. As in the crack problem, $\tilde{H}(z)$ can only have isolated singularities at the endpoints of Γ and is therefore entire. However, now the weaker condition $w = O(z^{-1})$ as $z \to \infty$ implies that $\tilde{H}(z) = O(1)$ as $z \to \infty$, so $\tilde{H}(z)$ is constant by Liouville's theorem (in contrast to the crack problem). Finally, we ensure that w is finite as the trailing edge $z = c$ by setting

$$\tilde{H}(z) = \tilde{H}(c) = -\frac{1}{\pi}\int_0^c \frac{g(\xi)\xi^{1/2}(c-\xi)^{1/2}}{\xi - z}\,\mathrm{d}\xi\bigg|_{z=c}, \tag{4.47}$$

giving

$$\begin{aligned} w(z) &= \frac{1}{\pi z^{1/2}(c-z)^{1/2}}\int_0^c g(\xi)\xi^{1/2}(c-\xi)^{1/2}\left(\frac{1}{\xi - z} - \frac{1}{\xi - c}\right)\mathrm{d}\xi \\ &= \frac{(c-z)^{1/2}}{\pi z^{1/2}}\int_0^c \frac{g(\xi)\xi^{1/2}}{(c-\xi)^{1/2}(\xi - z)}\,\mathrm{d}\xi. \end{aligned} \tag{4.48}$$

It is an exercise in perturbation methods to verify that the solution (4.48) satisfies $w(z) = \mathrm{O}(1)$ as $z \to c$. Equation (4.48) could have been obtained more directly by choosing $\tilde{w}(z) = (c-z)^{1/2}/z^{1/2}$, thereby incorporating the specified behaviour of $w(z)$ near the end points.

4.6 General Hilbert problem

We have seen that when $w_+ - w_-$ is given on Γ we can solve immediately for f and therefore for w. When $w_+ + w_-$ is given on Γ, we find a singular integral equation for f, but we can find w (and f) by introducing $\tilde{w}$ such that $\tilde{w}_+ = -\tilde{w}_- \neq 0$ on Γ. What about more general relations between w_+ and w_- on Γ?

The general so-called *Hilbert problem* is

$$a(z)w_+(z) + b(z)w_-(z) = c(z) \quad \text{on } \Gamma, \tag{4.49}$$

with a, $b \neq 0$ and c prescribed on Γ. Suppose we can find $\tilde{w}(z)$ holomorphic and non-zero away from Γ, with

$$a(z)\tilde{w}_+(z) = -b(z)\tilde{w}_-(z) \neq 0 \quad \text{on } \Gamma. \tag{4.50}$$

Then $W(z) = w(z)/\tilde{w}(z)$ satisfies

$$\begin{aligned} W_+(z) - W_-(z) &= \frac{w_+(z)}{\tilde{w}_+(z)} - \frac{w_-(z)}{\tilde{w}_-(z)} \\ &= \frac{w_+(z)}{\tilde{w}_+(z)} - \frac{w_-(z)}{-a(z)\tilde{w}_+(z)/b(z)} \\ &= \frac{a(z)w_+(z) + b(z)w_-(z)}{a(z)\tilde{w}_+(z)} \\ &= \frac{c(z)}{a(z)\tilde{w}_+(z)} \quad \text{on } \Gamma, \end{aligned} \tag{4.51}$$

giving

$$W(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{c(\zeta)}{a(\zeta)\tilde{w}_+(\zeta)(\zeta - z)} \, \mathrm{d}\zeta + H(z), \tag{4.52}$$

where $H(z)$ is an arbitrary function of z that is holomorphic away from the endpoints of Γ.

To solve for $\tilde{w}(z)$ we again take logs. Since $\tilde{w}_+(z)/\tilde{w}_-(z) = -b(z)/a(z)$, we get

$$\log \tilde{w}_+(z) - \log \tilde{w}_-(z) = \log \left(-\frac{b(z)}{a(z)} \right) \quad \text{on } \Gamma. \tag{4.53}$$

We can therefore use the Plemelj formulae as before to solve for $\tilde{w}(z)$ and hence find $w(z)$.

The general linear Cauchy singular integral equation for f:

$$a(z)f(z) + b(z) ⨍_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = c(z), \tag{4.54}$$

can be rewritten as a Hilbert problem for

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{\zeta - z} \mathrm{d}\zeta,$$

using the Plemelj formulae, and hence solved by following the above strategy.