Sheet 0: Revision of core complex analysis

Q1 (a) By treating $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$ as independent variables and using the chain rule write

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} \qquad \text{in terms of} \qquad \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \overline{z}}.$$

(b) Show that the Cauchy–Riemann equations for a holomorphic function $f(z, \overline{z}) = u(x, y) + \mathrm{i}v(x, y)$ (with u, v real) are equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

and that we then have

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{1}{\mathrm{i}}\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}.$$

(c) Calculate $\partial \overline{f(z)}/\partial z$ and deduce that $\overline{f}(z) = \overline{f(\overline{z})}$ is holomorphic. Show also that

$$\frac{\partial \overline{f(z)}}{\partial \overline{z}} = \overline{f'(z)}.$$

(d) Finally, show that Laplace's equation $u_{xx} + u_{yy} = 0$ is equivalent to $4u_{z\overline{z}} = 0$, and deduce that any real solution of Laplace's equation may be written in the form

$$u(x, y) = f(z) + \overline{f(z)}$$

for some holomorphic function $f(z)$.

Q2 (a) Show that, if $\zeta^2 = z^2 + 1$ and $z = \mathrm{i} + r_1 \mathrm{e}^{\mathrm{i}\theta_1} = -\mathrm{i} + r_2 \mathrm{e}^{\mathrm{i}\theta_2}$, where $r_1,\ r_2 > 0$, $\theta_1,\ \theta_2 \in \mathbb{R}$, then

$$\zeta = \pm (r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)/2}.$$

Explain why $z = \pm\mathrm{i}$ are the branch points of the multifunction $f(z) = (z^2 + 1)^{1/2}$.

(b) Consider the branch of $f(z) = (r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)/2}$, with $-\pi/2 < \theta_1,\ \theta_2 \leq 3\pi/2$. State the values of $(\theta_1 + \theta_2)/2$ on either side of the imaginary axis, and hence compute $f(\pm 0 + \mathrm{i}y)$ in terms of y for $y \in \mathbb{R}$. Deduce that the branch cut is $S = \{x + \mathrm{i}y : x = 0,\ |y| \leq 1\}$. Show that $f(z) \sim z$ as $|z| \to \infty$ (*i.e.* $f(z)/z \to 1$ as $|z| \to \infty$). Sketch the image of the cut z-plane $\mathbb{C}\backslash S$ under the map $\zeta = f(z)$.

(c) Consider the branch of $f(z) = +(r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)/2}$, with $-3\pi/2 < \theta_1 \leq \pi/2$ and $-\pi/2 < \theta_2 \leq 3\pi/2$. State the values of $(\theta_1 + \theta_2)/2$ on either side of the imaginary axis, and hence compute $f(\pm 0 + \mathrm{i}y)$ in terms of y for $y \in \mathbb{R}$. Deduce that the branch cut is along the imaginary axis from $z = -\mathrm{i}\infty$ to $z = -\mathrm{i}$ and from $z = \mathrm{i}$ to $z = \mathrm{i}\infty$. Compute $f(x)$ in terms of x for $x \in \mathbb{R}$. Show that

$$f(z) \sim \begin{cases} z & \text{as } |z| \to \infty \text{ with } \mathrm{Re}(z) > 0, \\ -z & \text{as } |z| \to \infty \text{ with } \mathrm{Re}(z) < 0. \end{cases}$$

Sketch the images of the half-planes $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(z) < 0$ under the map $\zeta = f(z)$.

Q3 Use contour integration to evaluate

$$\int_{-1}^{1} \frac{\sqrt{1 - x^2}}{x^2 + 1}\, \mathrm{d}x.$$

Hint *Consider the integral of* $\left(z^2 - 1\right)^{1/2} / \left(z^2 + 1\right)$, *with a branch cut from* $z = -1$ *to* $z = 1$, *around a large circle.*

Q4 Find the Fourier transforms of (a) $\mathrm{e}^{-|x|}$ and (b) $\mathrm{e}^{-a^2 x^2}$ (for the latter you may assume that $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$). Verify the inverses by contour integration, stating clearly where you use analytic continuation.

Hint (b): Integrate round a rectangular contour.