Sheet 0: Revision of core complex analysis

Q1

(a) Treating $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$ as independent variables and using the chain rule, we find

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial y}\frac{\partial}{\partial \bar{z}} = \mathrm{i}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

giving

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

(b) Hence, if $f = u + \mathrm{i}v$, with u, v real, then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)(u + \mathrm{i}v) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

so that

$$\frac{\partial f}{\partial \bar{z}} = 0$$

if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

which are the Cauchy-Riemann equations. Since

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)(u + \mathrm{i}v) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

we obtain

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} = -\mathrm{i}\frac{\partial f}{\partial y}$$

by the Cauchy-Riemann equations.

(c) Since $\overline{f(z)} = u - \mathrm{i}v$,

$$\frac{\partial \overline{f(z)}}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)(u - \mathrm{i}v) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0,$$

again by the Cauchy-Riemann equations. By replacing z with $\bar{z}$, we find $\partial\overline{f(\bar{z})}/\partial\bar{z} = 0$ so that $\bar{f}(z) = \overline{f(\bar{z})}$ is holomorphic. Similarly,

$$\frac{\partial \overline{f(z)}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)(u - \mathrm{i}v) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial v}{\partial x} = \overline{f'(z)}.$$

(d) It follows from (a) that

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)^2 = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2},$$
$$\frac{\partial^2}{\partial y^2} = \mathrm{i}^2\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)^2 = -\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\partial^2}{\partial \bar{z}^2},$$

giving

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z \partial \bar{z}}.$$

Integrating with respect to $\bar{z}$ and z gives

$$u_z = g'(z), \qquad\qquad u = g(z) + h(\bar{z}),$$

where g and h are arbitrary holomorphic functions of their arguments. Since u is real, $u \equiv \overline{u}$, leading to

$$\begin{aligned} & g(z) + h(\overline{z}) \equiv \overline{g(z)} + \overline{h(\overline{z})} \\ \Rightarrow \quad & g(z) + h(\overline{z}) \equiv \overline{g}(\overline{z}) + \overline{h}(z) \\ \Rightarrow \quad & g(z) - \overline{h}(z) \equiv \overline{g}(\overline{z}) - h(\overline{z}), \end{aligned}$$

where we again use the definition of the conjugate function $\overline{f}(z) = \overline{f(\overline{z})}$. Since this has to hold for all z and $\overline{z}$, which are independent variables, it follows that the left- and right-hand sides must both be a real constant, C say. Therefore our equation for u becomes

$$u = g(z) + \overline{g}(\overline{z}) - C = f(z) + \overline{f(z)}, \qquad \text{where} \quad f(z) = g(z) - \frac{C}{2}.$$

Q2

(a) If $\zeta = R\mathrm{e}^{\mathrm{i}\Theta}$, with R and Θ real, then

$$R^2\mathrm{e}^{2\mathrm{i}\Theta} = \zeta^2 = z^2 + 1 = (z - \mathrm{i})(z + \mathrm{i}) = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)},$$

giving

$$R = (r_1 r_2)^{1/2}, \qquad \Theta = \frac{1}{2}(\theta_1 + \theta_2) + \pi k\mathrm{i},$$

where $k \in \mathbb{Z}$. Thus

$$\zeta = \zeta_k = \pm(r_1 r_2)^{1/2}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2} \qquad (k \in \mathbb{Z}),$$

where $\pm = (-1)^k$. Since the jumps

$$\big[\theta_1\big]_{|z-a|=R} = \begin{cases} 0 & \text{for } a \neq \mathrm{i} \\ 2\pi & \text{for } a = \mathrm{i}, \end{cases} \qquad \big[\theta_2\big]_{|z-a|=R} = \begin{cases} 0 & \text{for } a \neq -\mathrm{i} \\ 2\pi & \text{for } a = -\mathrm{i}, \end{cases}$$

for all $R > 0$, ζ_k is continuous on $\partial D(a, \epsilon) := \{z \in \mathbb{C} : |z - a| = \epsilon\}$ as $\epsilon \to 0$ unless $a = \pm\mathrm{i}$, which are the only branch points because ζ_k is continuous on $\partial D(0, R)$ as $R \to \infty$.

(b) By definition (alternatively sketch values on the imaginary axis in the z-plane)

$$\frac{\theta_1 + \theta_2}{2} = \begin{cases} \pi/2 & \text{for } z = \pm 0 + \mathrm{i}y, \ y > 1, \\ 0 & \text{for } z = +0 + \mathrm{i}y, \ |y| < 1, \\ \pi & \text{for } z = -0 + \mathrm{i}y, \ |y| < 1, \\ -\pi/2 & \text{for } z = +0 + \mathrm{i}y, \ y < -1, \\ 3\pi/2 & \text{for } z = -0 + \mathrm{i}y, \ y < -1, \end{cases}$$

and $(r_1 r_2)^{1/2} = |y^2 - 1|^{1/2}$ on the imaginary axis, giving

$$f(\pm 0 + \mathrm{i}y) = \begin{cases} \mathrm{i}(y^2 - 1)^{1/2} & \text{for } y > 1, \\ \pm(1 - y^2)^{1/2} & \text{for } |y| < 1, \\ -\mathrm{i}(y^2 - 1)^{1/2} & \text{for } y < -1. \end{cases}$$

Hence, this branch of $f(z)$ is continuous across the imaginary axis except across the branch cut $S = \{x + \mathrm{i}y : x = 0, \ |y| \leq 1\}$. Note that this branch is holomorphic on $\mathbb{C}\backslash S$. Note that writing $(z^2+1)^{1/2} = (z-\mathrm{i})^{1/2}(z+\mathrm{i})^{1/2}$ shows that the selected branch is the same as choosing the branch of $(z \mp \mathrm{i})^{1/2}$ that is real and positive on the positive real axis with a cut on the segment $(-\infty\mathrm{i}, \pm\mathrm{i}]$ of the imaginary axis: the branches cuts annihilate each other on the segment $(-\infty\mathrm{i}, -\mathrm{i})$ of the imaginary axis in the sense that the product of $(z-\mathrm{i})^{1/2}$ and $(z+\mathrm{i})^{1/2}$ is continuous there.

For the behaviour at infinity, let $\theta = \arg(z)$ where we choose the range for θ according to $-\pi/2 < \theta \leq 3\pi/2$. As $r := |z| \to \infty$, $\theta_{1,2} \sim \theta$ and $r_{1,2} \sim r$, giving

$$f(z) \sim \sqrt{r^2}\mathrm{e}^{\frac{\mathrm{i}}{2}(\theta+\theta)} = r\mathrm{e}^{\theta\mathrm{i}} = z.$$

It follows that the image of $\mathbb{C}\backslash S$ under the map $\zeta = f(z)$ is the whole of the ζ-plane with the segment of the real axis from $\zeta = -1$ to $\zeta = 1$ removed (this segment is the image of the branch cut S under $\zeta = f(z)$).

(c) By definition (alternatively sketch values on the imaginary axis in the z-plane)

$$\frac{\theta_1+\theta_2}{2} = \begin{cases} \pm\pi/2 & \text{for } z = \pm 0 + \mathrm{i}y,\ y > 1, \\ 0 & \text{for } z = \pm 0 + \mathrm{i}y,\ |y| < 1, \\ \mp\pi/2 & \text{for } z = \pm 0 + \mathrm{i}y,\ y < -1, \end{cases}$$

and $(r_1 r_2)^{1/2} = |y^2-1|^{1/2}$ on the imaginary axis, giving

$$f(\pm 0 + \mathrm{i}y) = \begin{cases} \pm\mathrm{i}(y^2-1)^{1/2} & \text{for } y > 1, \\ (1-y^2)^{1/2} & \text{for } |y| < 1, \\ \mp\mathrm{i}(y^2-1)^{1/2} & \text{for } y < -1. \end{cases}$$

Hence, this branch of $f(z)$ is continuous across the imaginary axis except across the branch cut $T = \{x + \mathrm{i}y : x = 0,\ |y| \geq 1\}$. Note that this branch is holomorphic on $\mathbb{C}\backslash T$. Note that writing $(z^2+1)^{1/2} = (z-\mathrm{i})^{1/2}(z+\mathrm{i})^{1/2}$ shows that the selected branch is the same as choosing the same branch for $(z+\mathrm{i})^{1/2}$ as in (b), but the branch of $(z-\mathrm{i})^{1/2}$ that is real and positive on the positive real axis with a cut on the segment $[\mathrm{i}, \mathrm{i}\infty)$ of the imaginary axis.

Since $\theta_1 + \theta_2 = 0$ and $(r_1 r_2)^{1/2} = (1+x^2)^{1/2}$ on the real axis, it follows that $f(x) = (1+x^2)^{1/2}$ for $x \in \mathbb{R}$.

For the behaviour at infinity we proceed as (b). Let $\theta = \arg(z)$, where $-\pi/2 < \theta \leq 3\pi/2$ as before. As $|z| \to \infty$, $r_{1,2} \sim |z|$ as before. In the case that $\mathrm{Re}(z) > 0$, *i.e.* $-\pi/2 < \theta < \pi/2$, we again have $\theta_{1,2} \sim \theta$ as $|z| \to \infty$. Hence, $f(x) \sim z$ as $|z| \to \infty$ with $\mathrm{Re}(z) > 0$. As $|z| \to \infty$ with $\mathrm{Re}(z) < 0$, *i.e.* $\pi/2 < \theta < 3\pi/2$, we still have $\theta_2 \sim \theta$, but since we defined θ_1 to be in the range $-3\pi/2 < \theta_1 < \pi/2$, we now have $\theta_1 \sim \theta - 2\pi$. Hence, $f(z) \sim -z$ as $|z| \to \infty$ with $\mathrm{Re}(z) < 0$.

It follows that the image of both $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(z) < 0$ under the map $\zeta = f(z)$ is the half plane $\mathrm{Re}(\zeta) > 0$ with the segment of the real axis from $\zeta = 0$ to $\zeta = 1$ removed (this segment is the image of S under $\zeta = f(z)$, while T is mapped onto the imaginary ζ-axis).

Q3

Consider the function $f(z) = \left(z^2-1\right)^{1/2}/(z^2+1)$ for $z \in \mathbb{C}$. We choose the branch cut for $\left(z^2-1\right)^{1/2}$ from $z = -1$ to $z = 1$ along the real axis and we take the sign of the square root such that $\sqrt{z^2-1} > 0$ for $z = x > 1$. Thus, $f(z)$ is holomorphic on $\mathbb{C}\backslash\{[-1,1], \pm\mathrm{i}\}$. Consider the integral of $f(z)$ over the closed contour Γ_2 illustrated in figure 1. By contour deformation and Cauchy's Residue Theorem, the contour Γ_2 may be deformed to a contour Γ_1 which just encloses the branch cut, provided we add the residues of the poles at $z = \pm\mathrm{i}$, that is

$$\oint_{\Gamma_2} f(z)\,\mathrm{d}z = \oint_{\Gamma_1} f(z)\,\mathrm{d}z + 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}}[f(z)] + 2\pi\mathrm{i}\,\mathrm{Res}_{z=-\mathrm{i}}[f(z)]. \tag{1}$$

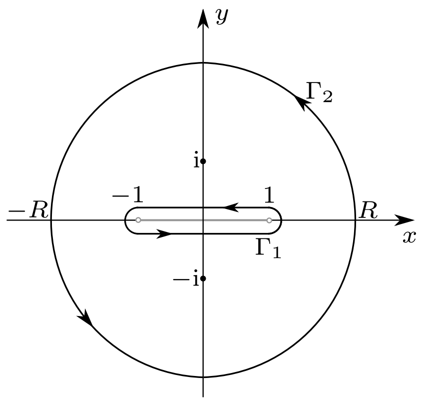


Figure 1: The two closed contours of integration

Note that our definition of the multifunction means that $\left(z^2-1\right)^{1/2} = \pm\mathrm{i}\sqrt{2}$ at $z = \pm\mathrm{i}$. Hence

$$\mathrm{Res}_{z=\mathrm{i}}[f(z)] = \mathrm{Res}_{z=-\mathrm{i}}[f(z)] = \frac{1}{\sqrt{2}}.$$

As $z \to \infty$, our choice of branch means that $f(z) \sim 1/z + O\left(1/z^2\right)$, so that

$$\lim_{R\to\infty} \oint_{\Gamma_2} f(z)\,\mathrm{d}z \sim \oint_{\Gamma_2} \left(\frac{1}{z} + O\left(z^{-2}\right)\right)\,\mathrm{d}z = 2\pi\mathrm{i}.$$

The integral around Γ_1 is found by integrating along the top of the branch cut, where $\left(z^2-1\right)^{1/2} = \mathrm{i}\left(1-x^2\right)$, and then along the bottom of the branch cut, where $\left(z^2-1\right)^{1/2} = -\mathrm{i}\left(1-x^2\right)$. (It can easily be verified that there is no contribution from the branch points $z = \pm 1$ themselves.) Therefore

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_1^{-1} \frac{\mathrm{i}\sqrt{1-x^2}}{x^2+1}\,\mathrm{d}x + \int_{-1}^{1} -\frac{\mathrm{i}\sqrt{1-x^2}}{x^2+1}\,\mathrm{d}x = -2\mathrm{i}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{x^2+1}\,\mathrm{d}x.$$

Finally plugging all the pieces into equation (1), we get

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{x^2+1}\,\mathrm{d}x = \pi\left(\sqrt{2}-1\right).$$

Q4

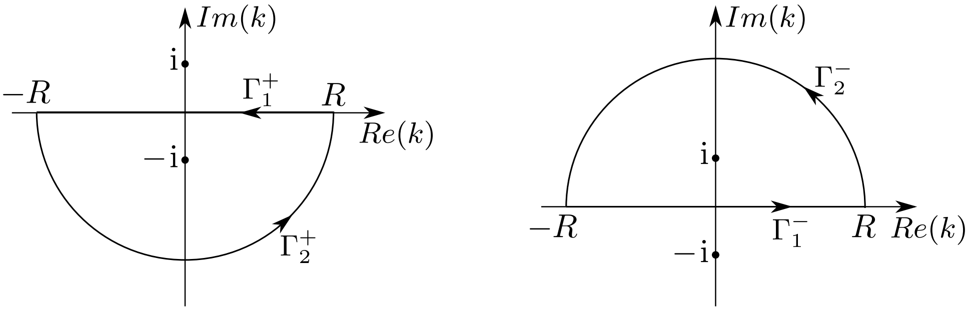
(a) The Fourier transform of $\mathrm{e}^{-|x|}$ is given by

$$\begin{aligned}\mathcal{F}\left(\mathrm{e}^{-|x|}\right) &= \int_{-\infty}^{\infty} \mathrm{e}^{-|x|}\mathrm{e}^{\mathrm{i}xk}\,\mathrm{d}x = \int_0^{\infty} \mathrm{e}^{-x+\mathrm{i}kx} + \mathrm{e}^{-x-\mathrm{i}kx}\,\mathrm{d}x \\ &= \left[\frac{\mathrm{e}^{-x+\mathrm{i}kx}}{-1+\mathrm{i}k} + \frac{\mathrm{e}^{-x-\mathrm{i}kx}}{-1-\mathrm{i}k}\right]_{x=0}^{\infty} = \frac{2}{1+k^2}.\end{aligned}$$

The integrals converge provided $\mathrm{Re}(-1+\mathrm{i}k) < 0$ and $\mathrm{Re}(-1-\mathrm{i}k) < 0$, *i.e.* $-1 < \mathrm{Im}(k) < 1$. We may use analytic continuation to extend to domain of definition to $\mathbb{C}\backslash\{\pm\mathrm{i}\}$. The inverse Fourier transform is given by

$$\mathcal{F}^{-1}\left(\frac{2}{1+k^2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k.$$

We evaluate this integral by closing the contour as illustrated in figure 2 and using the residue theorem. We close the contour either in the upper or the lower half-plane depending on the sign of x.



(a) Closed contour for $x > 0$. (b) Closed contour for $x < 0$.

Figure 2: Closing the contour for the inversion of the Fourier transform.

If $x > 0$, close the contour in the lower half-plane as illustrated in figure 2(a). Since $\mathrm{Re}(-\mathrm{i}kx) = \mathrm{Im}(k)x < 0$ for $k \in \Gamma_2^+$,

$$\lim_{R\to\infty} \int_{\Gamma_2^+} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k = 0$$

and the integral around $\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^+$ becomes

$$\lim_{R\to\infty} \oint_{\Gamma^+} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k = -\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k.$$

The function $f(k) = \mathrm{e}^{-\mathrm{i}kx}/(1+k^2)$ has two single poles at $k = \pm\mathrm{i}$. Only the pole at $k = -\mathrm{i}$ lies inside the closed contour Γ^+, so

$$\oint_{\Gamma^+} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k = 2\pi\mathrm{i}\,\mathrm{Res}_{k=-\mathrm{i}}\left[\frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\right] = -\pi\mathrm{e}^{-x}.$$

Combining these results then gives

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k = \mathrm{e}^{-x} \quad \text{for} \quad x > 0.$$

If $x < 0$, close the contour in the upper half-plane as illustrated in figure 2(b). Then $f(z)$ has one simple pole inside $\Gamma^- = \Gamma_1^- \cup \Gamma_2^-$ at $z = \mathrm{i}$, and $\mathrm{Re}(-\mathrm{i}kx) = \mathrm{Im}(k)x < 0$ on Γ_2^-. Similarly to the case in which $x > 0$, we find

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k = \lim_{R\to\infty}\frac{1}{2\pi}\oint_{\Gamma^-} \frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\,\mathrm{d}k = 2\mathrm{i}\,\mathrm{Res}_{k=\mathrm{i}}\left[\frac{\mathrm{e}^{-\mathrm{i}kx}}{1+k^2}\right] = \mathrm{e}^{x} \quad \text{for} \quad x < 0.$$

Combining the results for $x > 0$ and $x < 0$, we deduce that $\mathcal{F}^{-1}(2/(k^2+1)) = \mathrm{e}^{|x|}$, as required.

(b) The Fourier transform of $\mathrm{e}^{-a^2x^2}$, with $a > 0$, is given by

$$\begin{aligned}\mathcal{F}\left(\mathrm{e}^{-a^2x^2}\right) &= \int_{-\infty}^{\infty} \mathrm{e}^{-a^2x^2+\mathrm{i}kx}\,\mathrm{d}x \\ &= \mathrm{e}^{-k^2/(4a^2)}\int_{-\infty}^{\infty} \mathrm{e}^{-(ax-\mathrm{i}k/(2a))^2}\,\mathrm{d}x \\ &= \frac{1}{a}\mathrm{e}^{-k^2/4a^2}\int_{-\infty-\mathrm{i}\mathrm{Re}(k)/2a}^{\infty-\mathrm{i}\mathrm{Re}(k)/2a} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi \\ &= \frac{\sqrt{\pi}}{a}\mathrm{e}^{-k^2/4a^2}.\end{aligned}$$

The last equality is obtained by integrating the entire function $f(z) = \mathrm{e}^{-z^2}$ around the rectangle for which the vertical edges are given by $\mathrm{Re}(z) = \pm R$, with $R \gg 1$, and the horizontal edges by $\mathrm{Im}(z) = 0$, and $\mathrm{Im}(z) = -\mathrm{Re}(k)/2a$; since $f(z)$ is holomorphic inside the rectangle, the contour integral around the rectangle of $f(z)$ is equal to 0. Also, since $\mathrm{e}^{-z^2} \to 0$ as $\mathrm{Re}(z) \to \pm\infty$, the integrals of $f(z)$ along the vertical edges of the rectangle tend to 0 as $R \to \infty$. It follows that

$$\int_{-\infty-\mathrm{i}\mathrm{Re}(k)/2a}^{\infty-\mathrm{i}\mathrm{Re}(k)/2a} \mathrm{e}^{-z^2}\,\mathrm{d}z = \int_{-\infty}^{\infty} \mathrm{e}^{-z^2}\,\mathrm{d}z = \sqrt{\pi}.$$

Since this argument holds for all $k \in \mathbb{C}$ and the transform is entire, it is not necessary to use analytic continuation to derive the inverse Fourier transform, which is given by

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{\sqrt{\pi}}{a}\mathrm{e}^{-k^2/4a^2}\right) &= \frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-k^2/4a^2}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \frac{1}{2a\sqrt{\pi}}\mathrm{e}^{-a^2x^2}\int_{-\infty}^{\infty} \mathrm{e}^{-(k/2a+\mathrm{i}ax)^2}\,\mathrm{d}k \\ &= \frac{1}{\sqrt{\pi}}\mathrm{e}^{-a^2x^2}\int_{-\infty+\mathrm{i}ax}^{\infty+\mathrm{i}ax} \mathrm{e}^{-\kappa^2}\,\mathrm{d}\kappa = \mathrm{e}^{-a^2x^2}\end{aligned}$$

using the same arguments as above.