Sheet 1: More revision of core complex analysis and conformal mapping

Q1 Evaluate by contour integration:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^2}, \qquad \int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x, \qquad \int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta}.$$

(There is no need to prove the required estimates on integrals round large semicircles etc.)

Q2 (a) Find the image of the common part of the discs $|z-1|<1$ and $|z+\mathrm{i}|<1$ under the mapping $\zeta = 1/z$.

(b) Find the image of the strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$ under the map $\zeta = \sin z$.

(c) Find the image of the strip $-\pi < y < \pi$ under the map $\zeta = z + \mathrm{e}^z$.
[*Hint: where are the critical points of the map?*]

Q3 (a) D is the region exterior to the two circles $|z-1| = 1$ and $|z+1| = 1$. Find a conformal mapping from D to the exterior of the unit circle.
[*Hint: First apply an inversion with respect to the origin ($z \mapsto 1/z$), then rotate and scale so as to use the exponential function to get to a half plane, from where use Möbius.*

(b) Find a conformal map from the unit disc $|z| < 1$ onto the strip $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$, taking the origin to the origin, 1 to $\xi = +\infty$, -1 to $\xi = -\infty$. (Thus, the upper half of the unit circle maps to $\eta = \frac{\pi}{2}$ and the lower half to $\eta = -\frac{\pi}{2}$.)

(c) Find a conformal map of the quarter-disc $0 < |z| < 1$, $0 < \arg z < \frac{\pi}{2}$ to the upper half-plane $\eta > 0$, taking 0 to 0, 1 to 1, and i to ∞.