Sheet 2: Schwarz–Christoffel, boundary value problems, Free surface flows

Q1 Write down, as an integral, the Schwarz–Christoffel map from a half-plane to a rectangle, with the vertices being the images of the points $z = \pm 1$ and $z = \pm a$, where $a > 1$ is real. Explain why a cannot be specified arbitrarily, but is determined by the aspect ratio of the rectangle.

Q2 The domain D consists of the upper half-plane with a solid wall along the real axis. The segment of the imaginary axis from $z = 0$ to $z = \mathrm{i}$ is also impermeable to fluid. Find the complex potential for inviscid incompressible irrotational flow in D with velocity $(U_1, 0)$ at infinity.

Q3 The domain D consists of the right-hand half plane $x > 0$ with the circle $|z - a| = b$, $0 < b < a$, and its interior removed. Find the temperature $u(x, y)$ in steady heat flow if $u = 0$ on the y axis, $u = 1$ on $|z - a| = b$, and $u \to 0$ at infinity.
[*Hint: Show that the mapping* $\zeta = (z-\alpha)/(z+\alpha)$*, with* α *real and positive, takes* D *onto an annular region with the imaginary axis mapping to* $|\zeta| = 1$*. Show that, if* $\alpha^2 = a^2 - b^2$*, then the image of* D *is a concentric circular annulus.*]

Q4 (a) Carefully define a branch of the function $\cosh^{-1}(Z)$ that is holomorphic in the upper half-plane. What is $\cosh^{-1}(0)$? What is the derivative of $\cosh^{-1}(Z)$?

(b) Show that the Schwarz–Christoffel map from the upper half-plane to the exterior of the half-strip $0 < x < \infty$, $-1 < y < 1$ has the form

$$z = A + C\left(Z\sqrt{Z^2 - 1} - \cosh^{-1} Z\right),$$

and find the constants A and C.
[*Hint: Map* $Z = \pm 1$ *to the finite corners of the domain, and* $Z = \infty$ *to the vertex at* $x = \infty$]

(c) Hence find the complex potential $w(z)$ for potential flow past this obstacle with a uniform stream $(U_1, 0)$ at infinity.
[*Hint: Bearing in mind the behaviour of the mapping at infinity, think carefully about the potential in the* Z *plane: it is not 'constant* $\times$ Z*.'*]

Q5 Inviscid irrotational fluid flows steadily in the domain Ω shown in Figure 1, between a rigid wall ABC consisting of two semi-infinite straight line segments meeting at right angles, and a free surface $A'C'$.

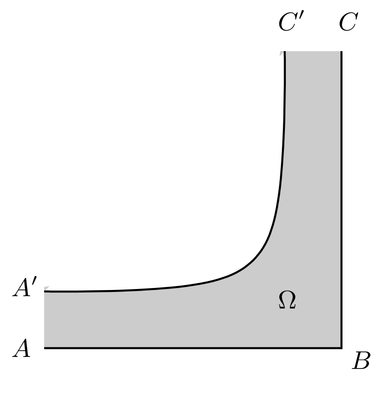

Figure 1: A jet climbing a wall.

The fluid layer has thickness 1 and velocity $(1, 0)$ far upstream, at AA'. The boundary value problem for the complex potential $w(z) = \phi + \mathrm{i}\psi$ is that $w(z)$ is holomorphic in Ω, with

$$\psi = 0 \text{ on } ABC, \qquad \psi = 1, \quad |w'| = 1 \text{ on } A'C',$$

where $w'(z) = u - iv$ is the complex velocity. In addition, take the reference point for ϕ so that $w = 0$ at B.

(a) Show that flow domain in the potential plane (w) is a strip, while in the hodograph plane (w') it is a quarter of the unit circle.

(b) Show that the map to a half plane is

$$\zeta = e^{\pi w} = \left(\frac{(w')^2 - 1}{(w')^2 + 1}\right)^2.$$

(c) Parametrise the free surface $A'C'$ by $w' = \mathrm{e}^{-\mathrm{i}\theta}$, where $0 \leq \theta \leq \pi/2$. Show that

$$\zeta = -\tan^2\theta, \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = \frac{2}{\pi}(\operatorname{cosec}\theta + \mathrm{i}\sec\theta) \qquad \text{on} \quad A'C'.$$

(d) Find parametric equations for the free surface from the real and imaginary parts of $\mathrm{d}z/\mathrm{d}\theta$. Check that it looks as it should.

Q6 A two-dimensional jet of inviscid irrotational fluid, of thickness $2h_\infty$ and moving to the right with speed 1, enters a semi-infinite rectangular cavity with walls at $y = \pm 1$ and $x = L$, as shown in Figure 2; the y axis is tangent to the free surface.

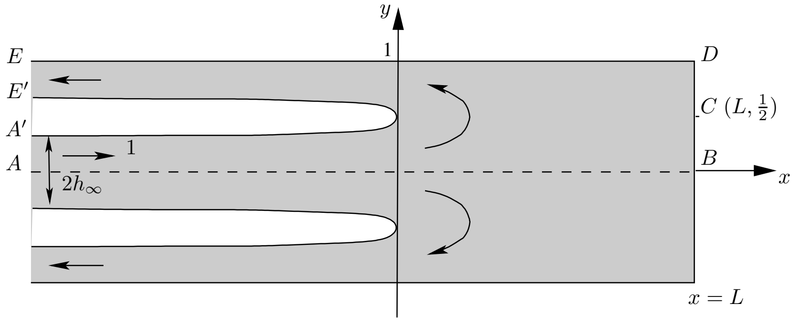

Figure 2: A jet entering a box.

The boundary value problem for the complex potential $w(z) = \phi + \mathrm{i}\psi$ for the upper half of the flow (within the strip $0 < y < 1$, $-\infty < x < L$) is that $w(z)$ is holomorphic in the fluid region, with

$$\psi = 0 \text{ on } ABCDE, \qquad \psi = h_\infty, \quad |w'| = 1 \text{ on } A'E',$$

where $w'(z) = u - \mathrm{i}v$ is the complex velocity. In addition, take the reference point for ϕ so that $w = 0$ at C.

(a) Sketch the flow domain in the potential and hodograph planes.

(b) Now consider the case $L = \infty$, with stagnant fluid far inside the cavity.

(i) Show that B, C and D coincide at the origin in the hodograph plane, so that the flow domain is the whole interior of the semicircle in the lower half plane.

(ii) Show that

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1 - \mathrm{e}^{\pi w/2h_\infty}}{1 + \mathrm{e}^{\pi w/2h_\infty}} = -\tanh\frac{\pi w}{4h_\infty}.$$

Find w satisfying $w = \mathrm{i}h_\infty$ at $z = \mathrm{i}/2$, the tip of the air finger shown in Figure 2.

(iii) Show that the free surface for this flow, $w = \phi + \mathrm{i}h_\infty$, $-\infty < \phi < \infty$, satisfies

$$\mathrm{e}^{-\pi x/2h_\infty}\cos\left(\frac{\pi(y - \frac{1}{2})}{2h_\infty}\right) = 1,$$

and show that $y \to \pm h_\infty$ as $x \to -\infty$ is only consistent if h_∞ takes a particular value, which you should find.