Sheet 3: Porous medium flow, Plemelj formulae

Q1 (a) The *harmonic moments* of a domain $D(t)$ are defined by

$$M_n(t) = \iint_{D(t)} z^n \, \mathrm{d}x\mathrm{d}y, \qquad n = 0, 1, 2, \ldots,$$

where $z = x + \mathrm{i}y$. What are the physical significance of $M_0(t)$ and $M_1(t)$?
If the boundary $\partial D(t)$ has outward normal velocity V_n, show (e.g. using Reynolds' Transport Theorem; see Part A Fluids & Waves) that

$$\frac{\mathrm{d}M_n}{\mathrm{d}t} = \oint_{\partial D} z^n V_n \, \mathrm{d}s.$$

(b) Use Green's Theorem on a region $R \subset \mathbb{R}^2$ to show that

$$\iint_R \frac{\partial G}{\partial \overline{z}}(z, \overline{z}) \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{2\mathrm{i}} \oint_{\partial R} G(z, \overline{z}) \, \mathrm{d}z$$

for any sufficiently smooth function $G(z, \overline{z})$. Deduce that

$$M_n(t) = \frac{1}{2\mathrm{i}} \oint_{\partial D} z^n \overline{z} \, \mathrm{d}z.$$

(c) The domain $D(t)$ is a saturated region of a porous medium, in which flow is driven by a point source of strength Q at $z = 0$. The potential $\phi(x, y, t)$ satisfies Laplace's equation in $D(t)$, with $\phi \sim (Q/2\pi) \log r$ at the origin (where $r^2 = x^2 + y^2$), together with $\phi = 0$, $\partial\phi/\partial n = V_n$ on $\partial D(t)$.
Use Green's Second Identity on $D(t)$ with a small circle around $z = 0$ removed to show that

$$\frac{\mathrm{d}M_0}{\mathrm{d}t} = Q, \qquad \frac{\mathrm{d}M_n}{\mathrm{d}t} = 0, \quad n > 0.$$

(d) The map $z = F(\zeta, t)$ maps the unit disc $|\zeta| < 1$ onto $D(t)$, with $F(0, t) = 0$. Show that

$$M_n(t) = \frac{1}{2\mathrm{i}} \oint_{|\zeta|=1} F(\zeta, t)^n \overline{F(\zeta, t)} \frac{\partial F}{\partial \zeta} \, \mathrm{d}\zeta.$$

Now suppose that $F(\zeta, t)$ is a polynomial of degree m, with coefficients $a_j(t)$, $j = 1 \ldots m$. Making use of the fact that $\overline{\zeta} = 1/\zeta$ on $|\zeta| = 1$, show that $M_n(t) = 0$ for $n \geq m$.
Hence find the nonzero moments for the quadratic map $F(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$, and crosscheck with the solution of the differential equations given in lectures.
Find formulae for M_0 and M_{m-1} for a general polynomial of degree m with complex coefficients.

[*Green's Theorem states that for any (suitably smooth) scalar functions $P(x, y)$ and $Q(x, y)$ and region $R \subset \mathbb{R}^2$ with (suitably smooth) boundary ∂R,*

$$\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \oint_{\partial R} (P \, \mathrm{d}y - Q \, \mathrm{d}x).$$

A corollary is Green's Second Identity:

$$\iint_R \left(u\nabla^2 v - v\nabla^2 u \right) \mathrm{d}x\mathrm{d}y = \oint_{\partial R} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}s,$$

again for suitably smooth $u(x, y)$ and $v(x, y)$.]

Q2 State the Plemelj formulae for the function defined by

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

where Γ is a contour in the complex plane.

By defining an appropriate branch of $w(z) = (z-1)^{\alpha-1}/(z+1)^\alpha$, where $0 < \alpha < 1$, and using the Plemelj formulae, evaluate

$$⨍_{-1}^{1} \frac{(1-t)^{\alpha-1}\,\mathrm{d}t}{(1+t)^\alpha (t-x)} \quad \text{for} \quad -1 < x < 1.$$

Q3 Many mechanics problems lead to the problem of finding $\phi(x, y)$ such that $\nabla^2\phi = 0$ except on $y = 0$, $0 \le x \le c$; $\lim_{y\downarrow\uparrow 0} \partial\phi/\partial y = g_\pm(x)$ for $0 < x < c$, where $g_\pm(x)$ is continuous on $0 \le x \le c$; $|\nabla\phi|$ is finite or has an inverse square-root singularity at $(0, 0)$ and $(c, 0)$; and $|\nabla\phi| \to 0$ as $x^2 + y^2 \to \infty$.

If $w(z) = -\mathrm{d}(\phi + \mathrm{i}\psi)/\mathrm{d}z$, where ψ is the harmonic conjugate of ϕ, then (I) w is holomorphic except on $\bar{\Gamma} = \{x + \mathrm{i}y : 0 \le x \le c,\ y = 0\}$; (II) $\operatorname{Im} w_+ = g_+$, $\operatorname{Im} w_- = g_-$ on the contour $\Gamma = \{x + \mathrm{i}y : 0 < x < c,\ y = 0\}$, where $g_\pm$ is continuous on $\bar{\Gamma}$; (III) w is finite or has an inverse square-root singularity at $z = 0$ and $z = c$; and (IV) $w \to 0$ as $z \to \infty$.

(a) Suppose $g_+(x) = -g_-(x)$. Use (I) and (II) to deduce that a possible solution is given by

$$w(z) = \frac{1}{\pi} \int_0^c \frac{g_+(\xi)\,\mathrm{d}\xi}{\xi - z} + h(z),$$

where $h(z)$ is an arbitrary function of z that is holomorphic on $\mathbb{C}\backslash\{0, c\}$ and real on Γ. Use (III), (IV) and Liouville's theorem to deduce that $h = 0$.

(b) Now suppose that $g_+(x) = g_-(x) = g(x)$. Show that, if $\widetilde{w}(z)$ is holomorphic and non-zero away from $\bar{\Gamma}$, with $\widetilde{w}_+(x) = -\widetilde{w}_-(x) \ne 0$ on Γ, then a possible solution for $w(z)$ is given by

$$\frac{w(z)}{\widetilde{w}(z)} = \frac{1}{\pi} \int_0^c \frac{g(\xi)\,\mathrm{d}\xi}{\widetilde{w}_+(\xi)(\xi - z)} + H(z),$$

where $H(z)$ is an arbitrary function of z holomorphic on $\mathbb{C}\backslash\{0, c\}$.

(c) An aerofoil problem has $g_+(x) = g_-(x) = -\alpha$, constant, together with the requirement that w has an inverse square-root singularity at $z = 0$, is finite at $z = c$, and $w \to 0$ as $z \to \infty$. By defining the branch of $\widetilde{w}(z) = (c - z)^{1/2} z^{-1/2}$ that has a cut along Γ and satisfies $\tilde{w}_+(\xi) > 0$ for $0 < \xi < c$, show that a possible solution is given by

$$w(z) = -\frac{\alpha\,(c - z)^{1/2}}{\pi z^{1/2}} \left[\int_0^c \frac{\xi^{1/2}\,\mathrm{d}\xi}{(c - \xi)^{1/2}\,(\xi - z)} + H(z) \right],$$

and determine $H(z)$.

(d) By relating the integral term to a suitable contour integral that can be deformed to a large circle, show that

$$w(z) = -\alpha \left(\frac{c - z}{z} \right)^{1/2} - i\alpha.$$

Noting that the behaviour of w at infinity is related to the circulation $\tilde{\Gamma}$ about the aerofoil by $w \sim \mathrm{i}\tilde{\Gamma}/2\pi z$, deduce that the circulation in this case is $\tilde{\Gamma} = -\pi c\alpha$.

Q4 Suppose f satisfies the Cauchy singular integral equation

$$a(t)f(t) + \frac{b(t)}{\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} = c(t) \qquad \text{on} \quad \Gamma, \tag{1}$$

where a, b and c are holomorphic in a neighbourhood of Γ.

(a) Show that, if

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

then $(a+b)w_+ + (b-a)w_- = c$ on Γ.

(b) Now suppose $a+b$ and $a-b$ are not zero on Γ, and that $\tilde{w}$ is holomorphic and non-zero away from Γ and that $(a+b)\tilde{w}_+ = -(b-a)\tilde{w}_- \neq 0$ on Γ. Show that

$$\left(\frac{w}{\tilde{w}}\right)_+ - \left(\frac{w}{\tilde{w}}\right)_- = \frac{c}{(a+b)\tilde{w}_+} \qquad \text{on} \quad \Gamma.$$

(c) Hence show that

$$w(z) = \frac{\tilde{w}(z)}{2\pi \mathrm{i}} \int_\Gamma \frac{c(\zeta)\,\mathrm{d}\zeta}{(a(\zeta)+b(\zeta))\tilde{w}_+(\zeta)(\zeta - z)}$$

is a possible solution for $w(z)$ and that (1) is satisfied by

$$f(t) = -\frac{b(t)\tilde{w}_+(t)}{(a(t)-b(t))} \frac{1}{\pi \mathrm{i}} \fint_\Gamma \frac{c(\zeta)}{(a(\zeta)+b(\zeta))\tilde{w}_+(\zeta)} \frac{\mathrm{d}\zeta}{\zeta - t} + \frac{c(t)a(t)}{a(t)^2 - b(t)^2}.$$