Sheet 4: Transforms, Wiener-Hopf

Q1 Suppose $f(x) = \mathrm{e}^{|x|}$ for $-\infty < x < \infty$.

(a) If $f(x) = f_+(x) + f_-(x)$, where $f_+(x) = 0$ for $x < 0$ and $f_-(x) = 0$ for $x > 0$, show that the Fourier transforms of f_+ and f_- are given by

$$\bar{f}_+(k) = \int_{-\infty}^{\infty} f_+(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = \frac{\mathrm{i}}{k - \mathrm{i}} \quad \text{for} \quad \mathrm{Im}(k) > 1$$

and

$$\bar{f}_-(k) = \int_{-\infty}^{\infty} f_-(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = -\frac{\mathrm{i}}{k + \mathrm{i}} \quad \text{for} \quad \mathrm{Im}(k) < -1$$

To which parts of the complex k-plane may $\bar{f}_\pm(k)$ be analytically continued?

(b) Use contour integration to evaluate

$$\frac{1}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_+(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k \quad \text{and} \quad \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\beta}^{\infty+\mathrm{i}\beta} \bar{f}_-(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k$$

for $x < 0$ and $x > 0$, where $\alpha > 1$ and $\beta < -1$.

(c) Over which part of the complex k-plane is it possible to define $\bar{f}(k)$? Sketch a suitable inversion contour Γ for which

$$f(x) = \frac{1}{2\pi}\int_\Gamma \bar{f}(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k.$$

Verify this result using contour integration.

Q2 (a) Show that

$$w(z) = \int_\Gamma g(\zeta)\mathrm{e}^{z\zeta}\,\mathrm{d}\zeta \qquad (*)$$

is a solution of Airy's equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + zw = 0$$

only if $g(\zeta) = A\mathrm{e}^{\zeta^3/3}$ and $[g(\zeta)\mathrm{e}^{z\zeta}]_\Gamma = 0$, where A is a constant. Identify two choices for Γ which lead to two independent solutions of the differential equation.

(b) Show that $(*)$ is a solution of Bessel's equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w = 0$$

only if $g(\zeta) = A/(1+\zeta^2)^{1/2}$ and $[(1+\zeta^2)g(\zeta)\mathrm{e}^{z\zeta}]_\Gamma = 0$. Identify two choices for Γ which lead to two independent solutions of the differential equation for $\mathrm{Re}(z) > 0$.

Q3 Suppose $w_+(z)$ is holomorphic in $Im(z) > \alpha$ and $w_-(z)$ is holomorphic in $Im(z) < \beta$, where $\alpha < \beta$, and neither w_+ nor w_- grows faster than a polynomial at infinity.

(a) Suppose $w_+(z) = w_-(z)$ in $\alpha < Im(z) < \beta$. Show that $w_+(z) = w_-(z)$ is a polynomial in z.

(b) Suppose $w_+(z) - w_-(z) = G(z)$, where G is holomorphic in $\alpha < Im(z) < \beta$ and $G \to 0$ as $|z| \to \infty$ in the strip. Show that $w_+(z) - G_+(z) = w_-(z) - G_-(z)$ is a polynomial, where

$$G_\pm(z) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\gamma_\pm}^{\infty+\mathrm{i}\gamma_\pm} \frac{G(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

for $\alpha < \gamma_+ < Im(z) < \gamma_- < \beta$.

(c) Suppose that $F(z)w_+(z) = w_-(z)$, where F is holomorphic and non-zero in $\alpha < Im(z) < \beta$ and $F \to 1$ as $|z| \to \infty$ in the strip. Show that $F_+w_+ = F_-w_-$ is a polynomial, where

$$F_\pm(z) = \exp\left(\frac{1}{2\pi \mathrm{i}} \int_{-\infty+\mathrm{i}\gamma_\pm}^{\infty+\mathrm{i}\gamma_\pm} \frac{\log\left(F(\zeta)\right) \mathrm{d}\zeta}{\zeta - z}\right) \qquad (\dagger)$$

for $\alpha < \gamma_+ < Im(z) < \gamma_- < \beta$.

(d) When $F(z) = (z^2 - a^2)/(z^2 - b^2)$, where a and b are distinct complex numbers with $\gamma = \min(Im(a), Im(b)) > 0$, verify *directly* that

$$F_+(z) = \frac{z+a}{z+b}, \qquad F_-(z) = \frac{z-b}{z-a} \qquad (\ddagger)$$

is a possible factorization.

(e) Suppose that

$$I_\pm(a) = \int_{-R+\mathrm{i}\gamma_\pm}^{R+\mathrm{i}\gamma_\pm} \frac{\log(\zeta - a)}{\zeta - z} \mathrm{d}\zeta$$

where a is as in (d) and $-\gamma < \gamma_+ < Im(z) < \gamma_- < \gamma$. Show that $\partial I_+/\partial a \to 0$, but that $\partial I_-/\partial a \to 2\pi\mathrm{i}/(z-a)$, as $R \to \infty$. Hence *deduce* ($\ddagger$) from ($\dagger$).

Q4 Suppose $u(x, y)$ satisfies the mixed boundary value problem

$$\nabla^2 u = \frac{\partial u}{\partial x} \quad \text{in } y > 0,$$

with

$$u = \mathrm{e}^{-ax} \quad \text{on } y = 0,\ x > 0, \qquad \frac{\partial u}{\partial y} = 0 \quad \text{on } y = 0,\ x < 0,$$

where $a > 0$, and $u \to 0$ as $x^2 + y^2 \to \infty$. Define

$$f_-(x) = \begin{cases} u(x,0) & x < 0, \\ 0 & x > 0, \end{cases} \qquad g_+(x) = \begin{cases} 0 & x < 0, \\ \dfrac{\partial u}{\partial y}(x,0) & x > 0. \end{cases}$$

Show that the Fourier transforms $\bar{f}_-(k)$ and $\bar{g}_+(k)$ of $f_-(x)$ and $g_+(x)$ respectively satisfy

$$\frac{1}{(k^2 - \mathrm{i}k)^{1/2}}\, \bar{g}_+(k) + \bar{f}_-(k) = -\frac{\mathrm{i}}{k + \mathrm{i}a},$$

where you should define precisely the branch of the multifunction $\left(k^2 - \mathrm{i}k\right)^{1/2}$. Deduce that

$$u(x,y) = \frac{1}{2\pi} \int_\Gamma \frac{\mathrm{i}(-\mathrm{i}a - \mathrm{i})^{1/2}}{(k + \mathrm{i}a)(k - \mathrm{i})^{1/2}}\, \mathrm{e}^{-y\left(k^2 - \mathrm{i}k\right)^{1/2} - \mathrm{i}kx}\, \mathrm{d}k,$$

where you should define precisely the branch of $(k - \mathrm{i})^{1/2}$, the appropriate value of $(-\mathrm{i}a - \mathrm{i})^{1/2}$ and a suitable inversion contour Γ.

[*You may assume without proof that* $k^{1/2}\bar{g}_+(k)$ *and* $k\bar{f}_-(k)$ *are bounded as* $k \to \infty$.]

Q5 Suppose $u(x, y)$ satisfies partial differential equation $\nabla^2 u = u$ in $y > 0$ subject to the mixed boundary conditions

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{for } x < 0, \qquad u(x, 0) = x \quad \text{for } x > 0,$$

and $u(x,y) \to 0$ as $y \to \infty$. By taking a Fourier transform and using the Wiener–Hopf method, show that

$$u(x,0) = \frac{1}{2\pi} \int_{\Gamma} A(k) e^{-ikx} \, dx, \qquad \text{where} \quad A(k) = -\frac{e^{-i\pi/4}(2+ik)}{2k^2(k-i)^{1/2}}, \qquad (\S)$$

giving precise definitions of $(k-i)^{1/2}$ and the integration contour Γ. [*You may assume without proof that the Fourier transforms of* $u(x,0)$ *and of* $\partial u/\partial y(x,0)$ *satisfy* $\bar{u}(k,0) = O\left(k^{-3/2}\right)$ *and* $\partial \bar{u}/\partial y(k,0) = O\left(k^{-1/2}\right)$ *as* $k \to \infty$.]

Verify that (§) gives the correct expression for $u(x,0)$ when $x > 0$.

Show that, for $x < 0$,

$$u(x,0) = \frac{1}{2\pi} \int_1^{\infty} \frac{(2-t)e^{tx}}{t^2\sqrt{t-1}} \, dt.$$