Sheet 4: Transforms, Wiener-Hopf

Q1

(i) Since $f_+(x) = 0$ for $x < 0$ and $f_+(x) = e^x$ for $x > 0$, the Fourier transform of f_+ is given by

$$\bar{f}_+(k) = \int_{-\infty}^{\infty} f_+(x)e^{\mathrm{i}kx}\,\mathrm{d}x = \int_0^{\infty} e^{(1+\mathrm{i}k)x}\,\mathrm{d}x = \left[\frac{e^{(1+\mathrm{i}k)x}}{1+\mathrm{i}k}\right]_0^{\infty} = \frac{-1}{1+\mathrm{i}k} = \frac{\mathrm{i}}{k-\mathrm{i}}$$

provided $e^{(1+\mathrm{i}k)x}$ vanishes as $x \to \infty$, *i.e.* provided k is such that $\mathrm{Re}(1+\mathrm{i}k) < 0$ or $\mathrm{Im}(k) > -1$. Similarly,

$$\bar{f}_-(k) = \int_{-\infty}^{\infty} f_-(x)e^{\mathrm{i}kx}\,\mathrm{d}x = \int_{-\infty}^{0} e^{(-1+\mathrm{i}k)x}\,\mathrm{d}x = \left[\frac{e^{(-1+\mathrm{i}k)x}}{-1+\mathrm{i}k}\right]_{-\infty}^{0} = \frac{1}{-1+\mathrm{i}k} = -\frac{\mathrm{i}}{k+\mathrm{i}}$$

provided $\mathrm{Im}(k) < -1$. Since $\bar{f}_\pm(k)$ has a pole at $k = \pm\mathrm{i}$ only, $f_+(k)$ can be analytically continued to $\mathbb{C}\backslash\{\mathrm{i}\}$, and $f_-(k)$ to $\mathbb{C}\backslash\{-\mathrm{i}\}$.

(ii) The inversion contour for $f_+(x)$ needs to be above any of the singularities of $\bar{f}_+(k)$. Since $\bar{f}_+(k)$ has a singularity at $k = \mathrm{i}$ only, we deduce from the inversion theorem that

$$\frac{1}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_+(k)e^{-\mathrm{i}kx}\,\mathrm{d}k = f_+(x)$$

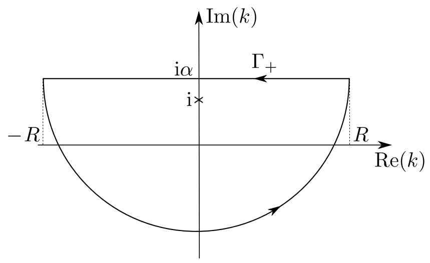
for $\alpha > 1$. We verify this result by contour integration as follows. For $x > 0$ close the contour at $k = -\mathrm{i}\infty$, as illustrated in (a) below. The integral over the semi-circle $|k-\mathrm{i}\alpha| = R$, $\mathrm{Im}(k-\mathrm{i}\alpha) < 0$ tends to zero as $R \to \infty$, giving

$$\frac{1}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_+(k)e^{-\mathrm{i}kx}\,\mathrm{d}k = \lim_{R\to\infty} -\frac{\mathrm{i}}{2\pi}\oint_\Gamma \frac{e^{-\mathrm{i}kx}}{k-\mathrm{i}}\,\mathrm{d}k = \mathrm{Res}_{k=\mathrm{i}}\left[\frac{e^{-\mathrm{i}kx}}{k-\mathrm{i}}\right] = e^x.$$

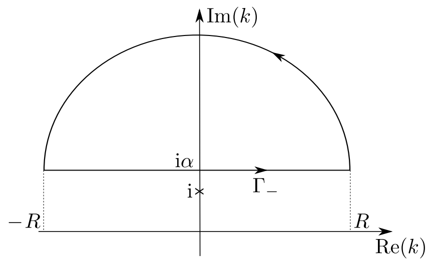
Similarly, for $x < 0$ close the contour at $k = \mathrm{i}\infty$, as illustrated in (b) below, to obtain

$$\frac{1}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_+(k)e^{-\mathrm{i}kx}\,\mathrm{d}k = \lim_{R\to\infty}\frac{\mathrm{i}}{2\pi}\oint_\Gamma \frac{e^{-\mathrm{i}kx}}{k-\mathrm{i}}\,\mathrm{d}k = 0,$$

since $f_+(k)$ doesn't have any singularities inside Γ.



(a) Inversion contour for $\bar{f}_+$ for $x > 0$



(b) Inversion contour for $\bar{f}_+$ for $x < 0$

The inversion contour for $f_-(x)$ needs to be below any of the singularities of $\bar{f}_-(k)$. Since $\bar{f}_-(k)$ has a simple pole at $z = -\mathrm{i}$ only, we deduce from the inversion theorem that

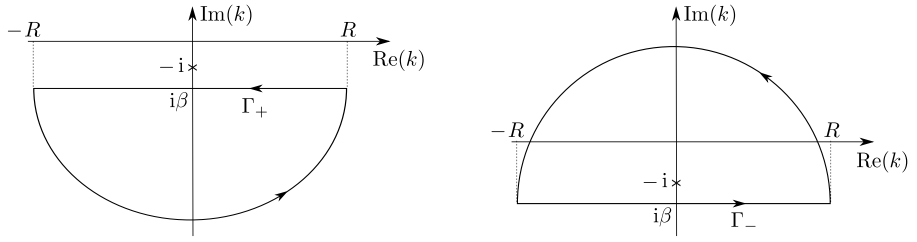
$$\frac{1}{2\pi}\int_{-\infty+\mathrm{i}\beta}^{\infty+\mathrm{i}\beta} \bar{f}_-(k)e^{-\mathrm{i}kx}\,\mathrm{d}k = f_-(x)$$

for $\beta < -1$. We verify this result by contour integration as before. We close the contour at $k = -\mathrm{i}\infty$ for $x > 0$ and at $k = \mathrm{i}\infty$ for $x < 0$, as illustrated below. For $x > 0$, we find

$$\frac{1}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_-(k)e^{-\mathrm{i}kx}\,\mathrm{d}k = \lim_{R\to\infty} -\frac{1}{2\pi}\oint_{\Gamma_+} \frac{-\mathrm{i}\,e^{-\mathrm{i}kx}}{k+\mathrm{i}}\,\mathrm{d}k = 0.$$

while for $x < 0$,

$$\frac{1}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_+(k)e^{-\mathrm{i}kx}\,\mathrm{d}k = \lim_{R\to\infty} \frac{1}{2\pi}\oint_{\Gamma_-} \frac{-\mathrm{i}\,e^{-\mathrm{i}kx}}{k+\mathrm{i}}\,\mathrm{d}k = \mathrm{Res}_{k=\mathrm{i}}\left[\frac{e^{-\mathrm{i}kx}}{k+\mathrm{i}}\right] = e^{-x}.$$



(a) Inversion contour for $\bar{f}_-$ for $x > 0$ (b) Inversion contour for $\bar{f}_-$ for $x < 0$

(iii) We define

$$\bar{f}(k) = \bar{f}_+(k) + \bar{f}_-(k) = \frac{\mathrm{i}}{k-\mathrm{i}} - \frac{\mathrm{i}}{k+\mathrm{i}} = \frac{-2}{k^2+1},$$

which is holomorphic on $\mathbb{C}\backslash\{-\mathrm{i},\mathrm{i}\}$. Since $\bar{f}_+$ is holomorphic on $\mathbb{C}\backslash\{\mathrm{i}\}$ and $\bar{f}_-$ is holomorphic on $\mathbb{C}\backslash\{-\mathrm{i}\}$), we can deform the inversion contours for $\bar{f}_+$ and $\bar{f}_-$ to the same contour Γ provided Γ passes above i and below $-$i, as illustrated below, so that

$$f(x) = f_-(x) + f_+(x) = \frac{1}{2\pi}\int_\Gamma \bar{f}_-(k)e^{-ikx}\,\mathrm{d}k + \frac{1}{2\pi}\int_\Gamma \bar{f}_+(k)e^{-ikx}\,\mathrm{d}k = \frac{1}{2\pi}\int_\Gamma \bar{f}(k)e^{-ikx}\,\mathrm{d}k.$$

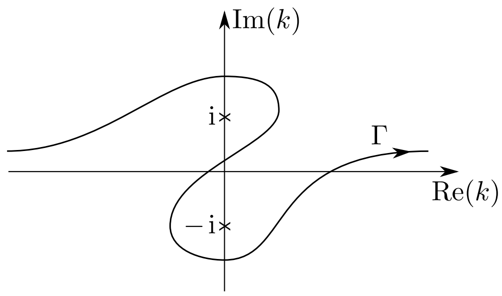


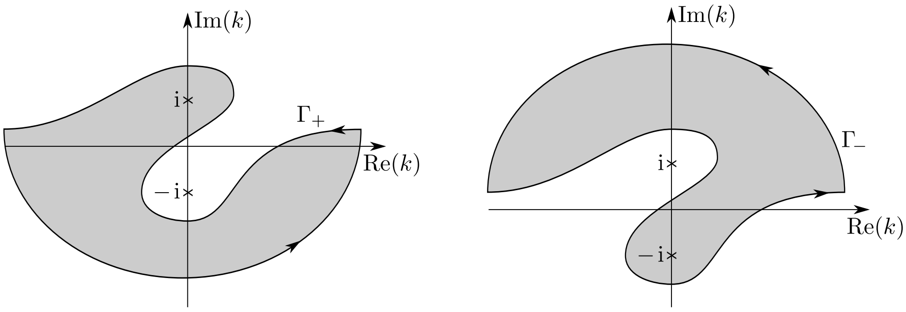
Figure 1: Inversion contour for $\bar{f}(k)$

NB: We can verify the inversion via contour integration. Close Γ at $-\mathrm{i}\infty$ for $x > 0$ and at $\mathrm{i}\infty$ for $x < 0$ (with a semi-circle of radius R), as illustrated below. For $x > 0$,

$$\frac{1}{2\pi}\int_\Gamma \bar{f}(k)e^{-k\mathrm{i}x}\,\mathrm{d}k = \lim_{R\to\infty} -\frac{1}{2\pi}\oint_{\Gamma_+} \frac{-2e^{-k\mathrm{i}x}}{k^2+1}\,\mathrm{d}k = -\mathrm{i}\,\mathrm{Res}_{k=\mathrm{i}}\left[\frac{-2e^{-k\mathrm{i}x}}{k^2+1}\right] = e^x,$$

while for $x < 0$

$$\frac{1}{2\pi}\int_\Gamma \bar{f}(k)e^{-k\mathrm{i}x}\,\mathrm{d}k = \lim_{R\to\infty} \frac{1}{2\pi}\oint_{\Gamma_-} \frac{-2e^{-k\mathrm{i}x}}{k^2+1}\,\mathrm{d}k = \mathrm{i}\,\mathrm{Res}_{k=-\mathrm{i}}\left[\frac{-2e^{-k\mathrm{i}x}}{k^2+1}\right] = e^{-x}.$$



(a) Closed contour for $\bar{f}$ for $x > 0$ (b) Closed contour for $\bar{f}$ for $x < 0$

Q2

(a) Differentiate under the integral sign to obtain

$$w'(z) = \int_\Gamma g(\zeta)\zeta e^{z\zeta}\,\mathrm{d}\zeta,$$
$$w''(z) = \int_\Gamma g(\zeta)\zeta^2 e^{z\zeta}\,\mathrm{d}\zeta,$$

Integrate by parts to find

$$zw(z) = \int_\Gamma zg(\zeta)e^{z\zeta}\,\mathrm{d}\zeta = \Big[g(\zeta)e^{z\zeta}\Big]_\Gamma - \int_\Gamma g'(\zeta)e^{z\zeta}\,\mathrm{d}\zeta,$$
$$zw''(z) = \int_\Gamma zg(\zeta)\zeta^2 e^{z\zeta}\,\mathrm{d}\zeta = \Big[g(\zeta)\zeta^2 e^{z\zeta}\Big]_\Gamma - \int_\Gamma \left(g'(\zeta)\zeta^2 + 2g(\zeta)\zeta\right)e^{z\zeta}\,\mathrm{d}\zeta.$$

It follows from Airy's equation that

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + zw = -\int_\Gamma \left(g'(\zeta) - \zeta^2 g(\zeta)\right)e^{\zeta z}\,\mathrm{d}\zeta + \Big[g(\zeta)e^{z\zeta}\Big]_\Gamma = 0.$$

Hence, Airy's equation is satisfied only if $g(\zeta)$ is such that

$$g'(\zeta) - \zeta^2 g(\zeta) = 0$$

for $\zeta \in \Gamma$ and the contour Γ is such that

$$\Big[g(\zeta)e^{z\zeta}\Big]_\Gamma = 0.$$

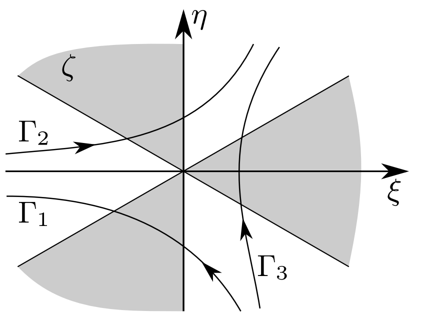
Since $g(\zeta) = C\exp(\zeta^3/3)$, where C is an arbitrary constant, the constraint on Γ becomes

$$\Big[Ce^{\zeta^3/3+z\zeta}\Big]_\Gamma = 0.$$

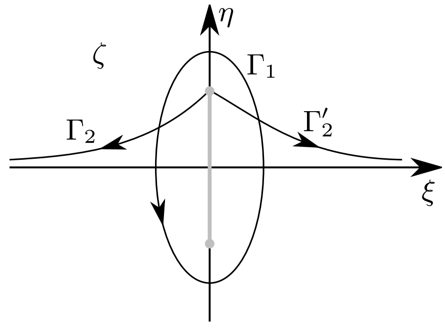
Thus, either Γ is a closed contour or $e^{\zeta^3/3+z\zeta}$ must be equal to zero at the end points of Γ. Since $e^{\zeta^3/3+z\zeta}$ is an entire function of ζ, the integral of this function over any closed contour in the ζ-plane will be equal to zero, which would give $w(z) = 0$ for all $z \in \mathbb{C}$. Therefore, for a non-trivial solution $w(z)$, we need that $e^{\zeta^3/3+\zeta z}$ to be equal to zero at the end points of Γ. Let $a_{1,2}$ denote the end points of Γ. Since $e^{az+a^3/3} \to 0$ iff $\mathrm{Re}(az + a^3/3) \to -\infty$, it follows that $a_{1,2}$ are at ∞ with

$$\frac{\pi}{6} + \frac{2k\pi}{3} < \arg(a_{1,2}) < \frac{\pi}{2} + \frac{2k\pi}{3},$$

where $k = 0$, 1 or 2. Note that if a_1 and a_2 lie in the same range, *e.g.* $\pi/6 < \arg(a_1) < \pi/2$ and $\pi/6 < \arg(a_2) < \pi/2$, we can close Γ at infinity, which would result in a trivial solution



(iii) Airy's equation



(iv) Bessel's equation

Figure 2: Contours of Integration for Airy's equation and Bessel's equation.

for w. Therefore the three possible contours are as illustrated below. Integrating along any two of these contours gives two linearly independent solutions for $w(z)$. The solution which follows from integrating along the third contour is a linear combination of these linearly independent solutions.

(b) Similarly, for Bessel's equation, we deduce that

$$z\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}+\frac{\mathrm{d}w}{\mathrm{d}z}+zw=-\int_\Gamma\left(g'(\zeta)(\zeta^2+1)+g(\zeta)\zeta\right)e^{z\zeta}\,\mathrm{d}\zeta+\left[g(\zeta)\left(\zeta^2+1\right)e^{z\zeta}\right]_\Gamma=0.$$

Hence, Bessel's equation is satisfied only if $g(\zeta)$ is such that

$$g'(\zeta)(\zeta^2+1)+g(\zeta)\zeta=0$$

for $\zeta\in\Gamma$ and the contour Γ is such that

$$\left[g(\zeta)\left(\zeta^2+1\right)e^{z\zeta}\right]_\Gamma=0.$$

Since

$$g(\zeta)=\frac{A}{\sqrt{\zeta^2+1}},$$

where A is an arbitrary constant and we choose the branch cut for $\sqrt{\zeta^2+1}$ from $\zeta=-\mathrm{i}$ to $\zeta=\mathrm{i}$ along the imaginary axis, the constraint on Γ becomes

$$\left[\sqrt{\zeta^2+1}\,e^{z\zeta}\right]_\Gamma=0.$$

Similar to before, we then have that Γ is either a closed contour or $\sqrt{\zeta^2+1}e^{z\zeta}$ disappears at the endpoints of Γ. Thus, a closed contour Γ_1 around the branch cut, as illustrated above, generates a solution valid for all $z\in\mathbb{C}$. Since $\sqrt{\zeta^2+1}=0$ for $\zeta=\pm i$ and $e^{z\zeta}\to 0$ as $\zeta\to\infty$ for $\mathrm{Re}(\zeta z)<0$, another valid choice is a contour Γ_2 that goes from $\zeta=\mathrm{i}$ to $\zeta=-\infty$ for $\mathrm{Re}(z)>0$ and a contour Γ_2' from $\zeta=\mathrm{i}$ to $\zeta=+\infty$ for $\mathrm{Re}(z)<0$, as illustrated above. Since Γ_1 cannot be deformed into Γ_2, these two contours generate two linearly independent solutions for $\mathrm{Re}(z)>0$.

Q3

This is all covered in the lecture notes.

Q4

Take the Fourier transform

$$\bar{u}(k,y) = \int_{-\infty}^{\infty} u(x,y)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$$

of the partial differential equation to obtain

$$\bar{u}_{yy} - (k^2 - \mathrm{i}k)\bar{u} = 0 \quad \text{in } y > 0.$$

Since $\bar{u}(k,y) \to 0$ as $y \to \infty$ by the far-field condition, the solution is

$$\bar{u}(k,y) = A(k)\mathrm{e}^{-(k^2-\mathrm{i}k)^{1/2}y},$$

where the branch of the square root must be chosen so that $\mathrm{Re}\left(k^2 - \mathrm{i}k\right)^{1/2} > 0$ on the inversion contour. Thus we choose a branch cut along the imaginary k-axis from $-\infty\mathrm{i}$ to 0 and from i to $\infty\mathrm{i}$, that is, with $\theta_1 = \arg(k)$ and $\theta_2 = \arg(k - \mathrm{i})$,

$$\left(k^2 - \mathrm{i}k\right)^{1/2} = k^{1/2}(k-\mathrm{i})^{1/2}, \quad k^{1/2} = |k|^{1/2}\mathrm{e}^{\mathrm{i}\theta_1/2}, \quad (k-\mathrm{i})^{1/2} = |k-\mathrm{i}|^{1/2}\mathrm{e}^{\mathrm{i}\theta_2/2},$$

with $-\pi/2 < \theta_1 < 3\pi/2$, $-3\pi/2 < \theta_2 < \pi/2$, so that $\mathrm{Re}\left(k^2 - \mathrm{i}k\right)^{1/2} > 0$ everywhere on the cut k-plane.

We write the boundary conditions on $y = 0$ as

$$u(x,0) = f_-(x) + \mathrm{H}(x)\mathrm{e}^{-ax}, \qquad \frac{\partial u}{\partial y}(x,0) = g_+(x),$$

where $\mathrm{H}(x)$ is the Heaviside function, and suppose that $g_+(x) = O\left(\mathrm{e}^{\alpha x}\right)$ as $x \to \infty$ and $f_-(x) = O\left(\mathrm{e}^{\beta x}\right)$ as $x \to -\infty$ for some constants α, β such that $\alpha < \beta$. Then $\bar{g}_+(k)$ is holomorphic in $\mathrm{Im}(k) > \alpha$ and $\bar{f}_-(k)$ is holomorphic in $\mathrm{Im}(k) < \beta$, so that both functions are holomorphic in the overlap strip $\alpha < \mathrm{Im}(k) < \beta$.

Since

$$\int_0^{\infty} \mathrm{e}^{-ax+\mathrm{i}kx}\,\mathrm{d}x = \frac{\mathrm{i}}{k + \mathrm{i}a}$$

for $\mathrm{Im}(k) > -a$, the boundary conditions on $y = 0$ give

$$\bar{u}(k,0) = \bar{f}_-(k) + \frac{\mathrm{i}}{k+\mathrm{i}a} \quad \text{for } -a < \mathrm{Im}(k) < \beta, \qquad \frac{\partial \bar{u}}{\partial y}(k,0) = \bar{g}_+(k) \quad \text{for } \mathrm{Im}(k) > \alpha,$$

provided $\beta > -a$, so that

$$A(k) = \bar{f}_-(k) + \frac{\mathrm{i}}{k+\mathrm{i}a}, \qquad -A(k)\left(k^2 - \mathrm{i}k\right)^{1/2} = \bar{g}_+(k).$$

Eliminating $A(k)$ gives

$$\frac{1}{(k^2-\mathrm{i}k)^{1/2}}\bar{g}_+(k) + \bar{f}_-(k) = -\frac{\mathrm{i}}{k+\mathrm{i}a}.$$

If $-a \le \alpha < \beta \le 1$, we can apply the Wiener-Hopf method.

Splitting $(k^2 - \mathrm{i}k)^{1/2}$, we have

$$\frac{\bar{g}_+(k)}{k^{1/2}} + (k-\mathrm{i})^{1/2}\bar{f}_-(k) = -\frac{\mathrm{i}(k-\mathrm{i})^{1/2}}{k+\mathrm{i}a}.$$

Splitting the right-hand side, we have

$$\frac{(k-\mathrm{i})^{1/2}}{k+\mathrm{i}a} = \frac{(k-\mathrm{i})^{1/2} - (-\mathrm{i}a-\mathrm{i})^{1/2}}{k+\mathrm{i}a} + \frac{(-\mathrm{i}a-\mathrm{i})^{1/2}}{k+\mathrm{i}a}, \tag{1}$$

where $(-\mathrm{i}a-\mathrm{i})^{1/2} = (1+a)^{1/2}\mathrm{e}^{-\mathrm{i}\pi/4}$ comes from evaluating $(k-\mathrm{i})^{1/2}$ at $k=-\mathrm{i}a$. The first-term on the right-hand side of (1) is holomorphic in the lower half-plane $\mathrm{Im}(k) < 1$, while the last term on the right-hand side is holomorphic in the upper half-plane $\mathrm{Im}(k) > -a$.

Hence,

$$\frac{\bar{g}_+(k)}{k^{1/2}} + \frac{\mathrm{i}(-\mathrm{i}a-\mathrm{i})^{1/2}}{k} = -(k-\mathrm{i})^{1/2}\bar{f}_-(k) - \frac{\mathrm{i}\left((k-\mathrm{i})^{1/2} - (-\mathrm{i}a-\mathrm{i})^{1/2}\right)}{k+\mathrm{i}a} \quad \text{for } \alpha < \mathrm{Im}(k) < \beta \tag{2}$$

with the left-hand side holomorphic in $\mathrm{Im}(k) > \alpha$ and the right-hand side holomorphic in $\mathrm{Im}(k) < \beta$. The right-hand side of (2) is the analytic continuation of the left-hand side of (2) into the lower half-plane, so together they define an entire function, $E(k)$ say. Since $k^{1/2}\bar{g}(k)$ and $k\bar{f}(k)$ are bounded at infinity, $E(k)$ tends to zero at infinity, so by Liouville's theorem, $E(k) \equiv 0$.

It follows that

$$A(k) = \bar{f}_-(k) + \frac{\mathrm{i}}{k+\mathrm{i}a} = \frac{\mathrm{i}(-\mathrm{i}a-\mathrm{i})^{1/2}}{(k+\mathrm{i}a)(k-\mathrm{i})^{1/2}},$$

giving

$$\bar{u}(k,y) = \frac{\mathrm{i}(-\mathrm{i}a-\mathrm{i})^{1/2}}{(k+\mathrm{i}a)(k-\mathrm{i})^{1/2}}\,\mathrm{e}^{-\left(k^2-\mathrm{i}k\right)^{1/2}y}.$$

Hence, the solution is given by

$$u(x,y) = \frac{1}{2\pi}\int_\Gamma \frac{\mathrm{i}(-\mathrm{i}a-\mathrm{i})^{1/2}}{(k+\mathrm{i}a)(k-\mathrm{i})^{1/2}}\,\mathrm{e}^{-\left(k^2-\mathrm{i}k\right)^{1/2}y-\mathrm{i}kx}\,\mathrm{d}k,$$

where analytic continuation of $\bar{u}(k,y)$ and the deformation theorem allow us to deform the inversion contour Γ out of the overlap strip $\alpha < \mathrm{Im}(k) < \beta$ provided we do not cross the branch cuts of $\bar{u}(k,y)$ (along the imaginary k-axis from $-\infty\mathrm{i}$ to 0 and from i to $\infty\mathrm{i}$).

Q5

Define

$$f_-(x) = \begin{cases} u(x,0) & x<0, \\ 0 & x>0, \end{cases} \qquad g_+(x) = \begin{cases} 0 & x<0, \\ \dfrac{\partial u}{\partial y}(x,0) & x>0, \end{cases}$$

so that $u(x,y)$ satisfies the boundary conditions

$$u(x,0) = f_-(x) + x\,\mathrm{H}(x), \qquad \frac{\partial u}{\partial y}(x,0) = g_+(x),$$

where $\mathrm{H}(x)$ is the Heaviside function. Note that

$$\int_{-\infty}^{\infty} x\,\mathrm{H}(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \int_0^\infty x\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = -\frac{1}{k^2}$$

for $\mathrm{Im}(k) > 0$. We suppose that $g_+(x) = O\left(\mathrm{e}^{\alpha x}\right)$ as $x \to +\infty$ and that $f_-(x) = O\left(\mathrm{e}^{\beta x}\right)$ as $x \to -\infty$, where $0 \le \alpha < \beta$. Then $\bar{g}_+(k)$ exists for $\mathrm{Im}(k) > \alpha$ and $\bar{f}_-(k)$ exists for $\mathrm{Im}(k) < \beta$, so that they are both defined on the overlap strip $\Omega = \{k \in \mathbb{C} : \alpha < \mathrm{Im}(k) < \beta\}$.

Now take the Fourier transform of the whole problem:

$$\frac{\partial^2 \bar{u}}{\partial y^2} = \left(k^2+1\right) u \qquad\qquad y>0$$

$$\bar{u} = \bar{f}_-(k) - \frac{1}{k^2}, \qquad \frac{\partial \bar{u}}{\partial y} = \bar{g}_+(k) \qquad\qquad y=0,$$

$$\bar{u} \to 0 \qquad\qquad y \to \infty,$$

on the overlap strip Ω. The general solution is

$$\bar{u}(k,y) = A(k)\mathrm{e}^{-y\left(k^2+1\right)^{1/2}},$$

where the square root must be defined to have positive real part on the inversion contour. Specifically, define

$$\left(k^2+1\right)^{1/2} = (k+1)^{1/2}(k-1)^{1/2},$$

where

$$(k+1)^{1/2} = |k+1|^{1/2}\exp\left(\frac{\mathrm{i}\arg(k+1)}{2}\right), \qquad (k-1)^{1/2} = |k-1|^{1/2}\exp\left(\frac{\mathrm{i}\arg(k-1)}{2}\right),$$

$$\arg(k+1) \in [-\pi/2, 3\pi/2), \qquad \arg(k-1) \in [-3\pi/2, \pi/2),$$

so the branch cut is along the imaginary axis, from $k=+\mathrm{i}$ to $+\mathrm{i}\infty$ and from $k=-\mathrm{i}\infty$ to $-\mathrm{i}$.

Plug in the boundary conditions to get

$$A(k) = \bar{f}_-(k) - \frac{1}{k^2}, \qquad -\left(k^2+1\right)^{1/2} A(k) = \bar{g}_+(k),$$

and elimination of $A(k)$ leads to the Wiener–Hopf problem

$$\bar{g}_+(k) + \left(k^2+1\right)^{1/2}\bar{f}_-(k) = \frac{\left(k^2+1\right)^{1/2}}{k^2} \quad \text{for } 0 \le \alpha < \mathrm{Im}(k) < \beta \le 1.$$

First split the singularities in $\left(k^2+1\right)^{1/2}$:

$$\frac{\bar{g}_+(k)}{(k+\mathrm{i})^{1/2}} + (k-\mathrm{i})^{1/2}\bar{f}_-(k) = \frac{(k-\mathrm{i})^{1/2}}{k^2}. \tag{3}$$

On the right-hand side, we need to split the pole at $k=0$ from the branch point at $k=\mathrm{i}$. Do this by Taylor expanding the numerator to eliminate the pole:

$$\frac{(k-\mathrm{i})^{1/2}}{k^2} = \frac{(-\mathrm{i})^{1/2}(1+\mathrm{i}k/2)}{k^2} + \frac{(k-\mathrm{i})^{1/2}-(-\mathrm{i})^{1/2}(1+\mathrm{i}k/2)}{k^2}.$$

Now in the second term on the right-hand side, the singularity at $k=0$ has been removed. Here $(-\mathrm{i})^{1/2}$ is $(k-\mathrm{i})^{1/2}$ evaluated at $k=0$, and therefore equal to $\mathrm{e}^{-\mathrm{i}\pi/4}$ by our choice of branch.

Thus we rearrange (3) to

$$\frac{\bar{g}_+(k)}{(k+\mathrm{i})^{1/2}} - \frac{(-\mathrm{i})^{1/2}(1+\mathrm{i}k/2)}{k^2} = -(k-\mathrm{i})^{1/2}\bar{f}_-(k) + \frac{(k-\mathrm{i})^{1/2}-(-\mathrm{i})^{1/2}(1+\mathrm{i}k/2)}{k^2}$$

for $0 \le \alpha < \mathrm{Im}(k) < \beta \le 1$, where the left-hand side is holomorphic in $\mathrm{Im}(k) > \alpha$ and the right-hand side is holomorphic in $\mathrm{Im}(k) < \beta$. Therefore the right-hand side is the analytic continuation of the left-hand side into the lower half-plane, and between them they define an entire function $E(k)$, say. By the hint, we know that the left- and right-hand sides both tend to zero in their respective half-planes, and it follows by Liouville's Theorem that $E(k) \equiv 0$.

We then solve for $\bar{f}_-(k)$ and use it to evaluate

$$\bar{u}(k,0) = A(k) = \bar{f}_-(k) - \frac{1}{k^2} = -\frac{e^{-i\pi/4}(2+ik)}{2k^2(k-i)^{1/2}},$$

so the inversion theorem gives

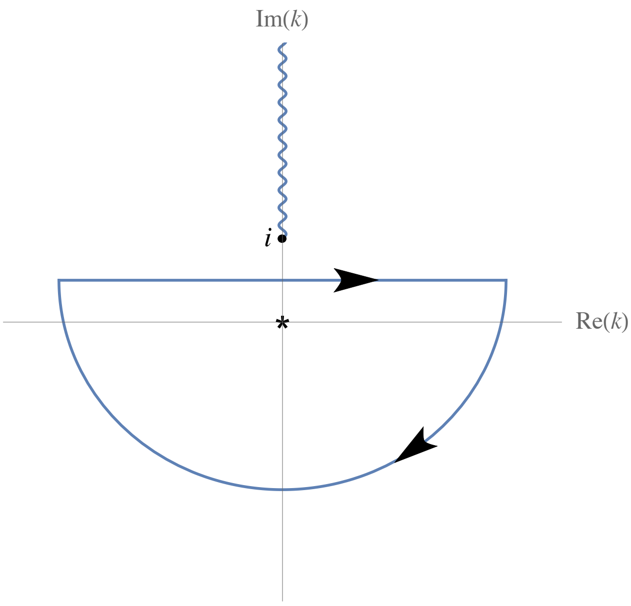
$$u(x,0) = -\frac{e^{-i\pi/4}}{4\pi}\int_\Gamma \frac{(2+ik)e^{-ikx}}{k^2(k-i)^{1/2}}\,dk,$$

where the integration contour Γ passes between the pole at $k=0$ and the branch point at $k=i$.

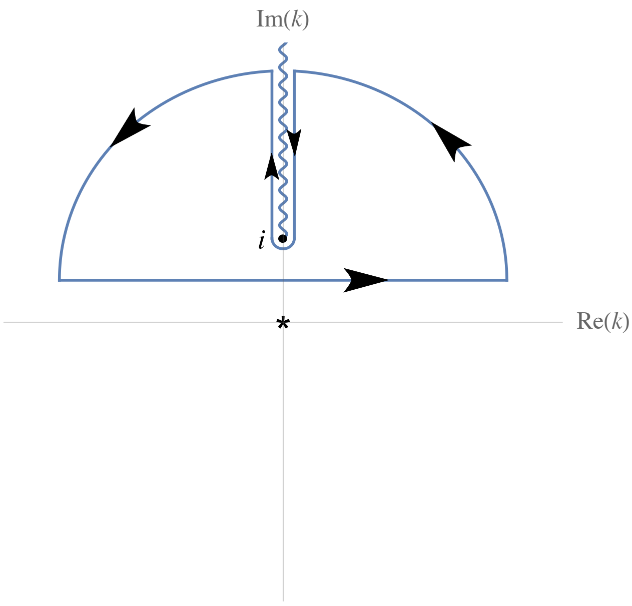
For $x>0$, we close the contour in the lower half k-plane as shown in (a) below, just picking up the residue from the pole at $k=0$:

$$\begin{aligned} u(x,0) &= -2\pi i\,\mathrm{res}\left[-\frac{e^{-i\pi/4}}{4\pi}\frac{(2+ik)e^{-ikx}}{k^2(k-i)^{1/2}}; k=0\right], \\ &= \frac{e^{i\pi/4}}{2}\,\mathrm{res}\left[\frac{(2+ik)e^{-ikx}}{k^2(k-i)^{1/2}}; k=0\right] \quad \text{for } x>0, \end{aligned} \tag{4}$$

where the minus sign comes from the clockwise sense of the integration contour.



(a) Closing the integration contour for $x>0$.



(b) Integration around the branch cut for $x<0$.

To calculate the residue, expand about $k=0$:

$$\begin{aligned} \frac{(2+ik)e^{-ikx}}{k^2(k-i)^{1/2}} &= \frac{2}{e^{-i\pi/4}}\frac{1}{k^2}\left(1+\frac{ik}{2}\right)(1+ik)^{-1/2}e^{-ikx} \\ &\sim \frac{2e^{i\pi/4}}{k^2}\left(1+\frac{ik}{2}\right)\left(1-\frac{ik}{2}+\cdots\right)(1-ikx+\cdots) \\ &\sim \frac{2e^{i\pi/4}}{k^2}(1-ikx+\cdots). \end{aligned}$$

Therefore the residue is $-2x\mathrm{e}^{3\mathrm{i}\pi/4}$, and (4) gives

$$u(x,0) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{2}\left(-2x\mathrm{e}^{3\mathrm{i}\pi/4}\right) = x \quad \text{for } x > 0,$$

as required.

For $x < 0$, we have to close the integration contour in the upper half-plane, integrating along the branch cut as shown in diagram (b) above. Just to the right of the branch cut, with $\mathrm{Re}(k) = 0_+$, we have $k = \mathrm{i}t$, where $t > 1$, and $(k - \mathrm{i})^{1/2} = \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{t-1}$. Just to the left of the branch cut, with $\mathrm{Re}(k) = 0_-$, we again have $k = \mathrm{i}t$, where $t > 1$, but now $(k-\mathrm{i})^{1/2} = \mathrm{e}^{-3\mathrm{i}\pi/4}\sqrt{t-1}$. Combining the two contributions, we get

$$\begin{aligned} u(x,0) &= -\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{4\pi}\left\{\int_1^\infty \frac{(2-t)\mathrm{e}^{xt}\mathrm{i}\,\mathrm{d}t}{-t^2\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{t-1}} - \int_1^\infty \frac{(2-t)\mathrm{e}^{xt}\mathrm{i}\,\mathrm{d}t}{-t^2\mathrm{e}^{-3\mathrm{i}\pi/4}\sqrt{t-1}}\right\} \\ &= \frac{1}{2\pi}\int_1^\infty \frac{(2-t)\mathrm{e}^{xt}}{t^2\sqrt{t-1}}\,\mathrm{d}t \quad \text{for } x < 0, \end{aligned}$$

as required.