2 Further conformal mapping

2.1 Introduction

Here we extend the ideas on conformal mapping introduced in the previous section. The Riemann Mapping Theorem guarantees that any simply connected domain D can be mapped onto the unit disc (for example). However, there is no general method to construct the required map for any given domain. One exception occurs if D is a polygon. The *Schwarz–Christoffel formula* in principle gives the conformal map from the upper half-plane to any given polygonal region. We will also show how conformal mapping can be used in practice in the solution of Laplace's equation.

2.2 Schwarz–Christoffel mapping

A (rare) constructive method for finding conformal maps (as opposed to cataloguing them) is the Schwarz–Christoffel formula. This lets us map a half-plane to a polygon (and there is an extension to circular polygons), and hence the inverse maps a polygon to a half-plane.

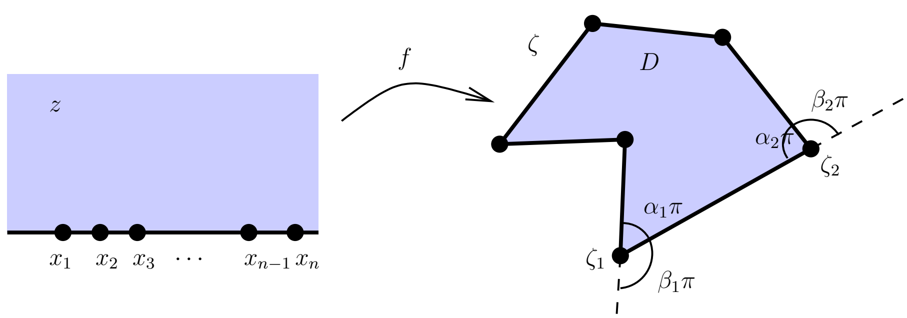


Figure 2.1: We seek a conformal map $z \mapsto \zeta = f(z)$ from the upper half-plane to a polygon with interior angles $\alpha_n\pi$ and corresponding exterior angles $\beta_n\pi$.

Our target domain is a polygon D with interior angles $\alpha_1\pi$,$\alpha_2\pi$, ... , $\alpha_n\pi$, at the vertices $\zeta = \zeta_1$, ζ_2, ... , ζ_n, as shown in Figure 2.1. These vertices are ordered so that increasing n means travelling round the polygon in the anticlockwise sense. We define

$$\beta_j\pi = \pi - \alpha_j\pi, \tag{2.1}$$

so that $\beta_j\pi$ is the exterior angle. Generally, $\beta_j > 0$ at a corner where we turn *left* and $\beta_j < 0$ at a corner where we turn *right*. Then the conditions

$$\sum_{j=1}^{n} \beta_j = 2, \qquad -2 \le \beta_j \le 2 \tag{2.2}$$

are necessary for the polygon to close. Now our aim is to find a mapping $\zeta = f(z)$ which maps the upper half-plane $y > 0$ onto D with the real axis mapping to ∂D and x_1, x_2, ... , x_n mapping to the vertices ζ_1, ζ_2, ... , ζ_n.

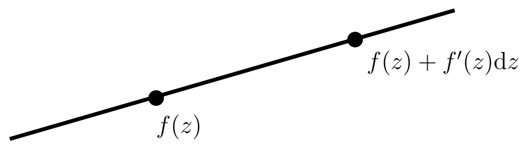


Figure 2.2: The direction of the tangent to ∂D is given by $\arg f'(z)$.

As shown schematically in Figure 2.2, the tangent to ∂D has direction angle $\arg f'(z)$, since $\mathrm{d}z = \mathrm{d}x$ is real on ∂D. This angle is supposed to be constant on each edge of the polygon ∂D. At x_j, the preimage of vertex j, the tangent angle increases by $\beta_j\pi$ and therefore we must have

$$\left[\arg f'(x)\right]_{x_j^-}^{x_j^+} = \beta_j\pi. \tag{2.3}$$

First consider the case of a single vertex, with pre-image at $z = x_j$. A function $f_j(z)$ such that

$$f_j'(z) = (z - x_j)^{-\beta_j} \tag{2.4}$$

(with a suitable branch defined) has the properties that

$$\arg f_j'(x) = \begin{cases} 0 & x > x_j, \\ -\beta_j\pi & x < x_j, \end{cases} \tag{2.5}$$

and therefore satisfies the jump condition (2.3). In addition, $f_j'(z) \neq 0$ for $z \neq x_j$, and therefore the resulting map is conformal away from the vertex.

When there are several vertices, the jump condition (2.3) is satisfied at each vertex by a *product* of functions of the form (2.4). If we try

$$f'(z) = C\prod_{j=1}^{n} f_j'(z), \tag{2.6}$$

where C is some constant, then

$$\arg f'(z) = \arg C + \sum_j \arg f_j'(z) \tag{2.7}$$

has exactly the right properties. Therefore a map from the upper-half plane to D is $\zeta = f(z)$, where

$$\frac{\mathrm{d}f}{\mathrm{d}z} = C\prod_{j=1}^{n}(z - x_j)^{-\beta_j}. \tag{2.8}$$

Hence

$$\zeta = f(z) = A + C\int^z \prod_{j=1}^{n}(t - x_j)^{-\beta_j}\,\mathrm{d}t, \tag{2.9}$$

where A and C fix the location and rotation/scaling of the polygon.

Notes

1. It can be shown that (2.9) is a one-to-one map from $\operatorname{Im} z > 0$ to D.

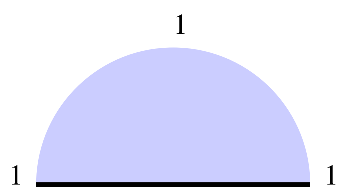
2. We are allowed by the Riemann Mapping Theorem to fix the pre-images of 3 boundary points, i.e. 3 of the x_j. Any more have to be found as part of the solution (by solving $f(x_j) = \zeta_j$).

3. We can choose one of the x_j to be at infinity. If (without loss of generality) $x_n = \infty$, then

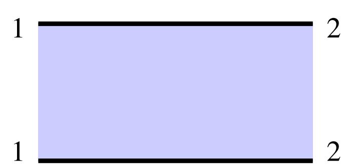
$$f(z) = A + C \int^z \prod_{j=1}^{n-1} (t - x_j)^{-\beta_j} \, \mathrm{d}t. \tag{2.10}$$

4. The definition of a polygon is elastic: it includes those with vertices at ∞ and those with interior angles of 2π. Some examples are shown in Figure 2.3.

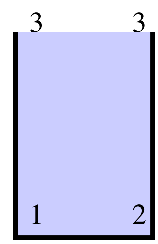
5. Most tractable examples are degenerate (*e.g.* they have a vertex at ∞) and use symmetry to simplify the integration.



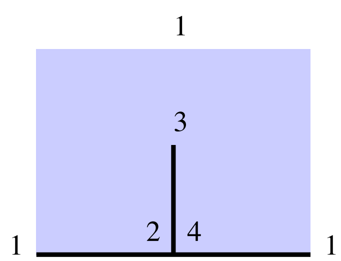
(a) $n = 1$, $\alpha_1 = -1$, $\beta_1 = 2$



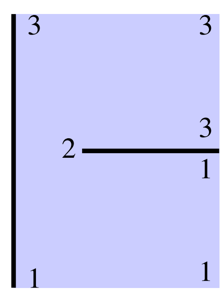
(b) $n = 2$, $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$



(c) $n = 3$, $\alpha_1 = \alpha_2 = 1/2$, $\alpha_3 = 0$, $\beta_1 = \beta_2 = 1/2$, $\beta_3 = 1$



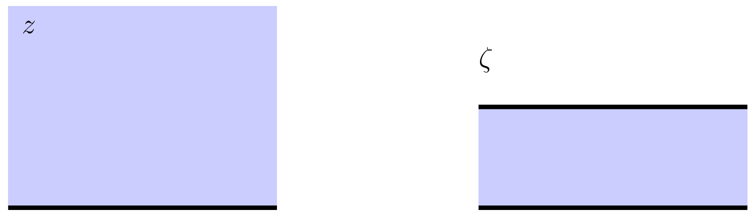
(d) $n = 4$, $\alpha_1 = -1$, $\alpha_2 = \alpha_4 = 1/2$, $\alpha_3 = 2$, $\beta_1 = 2$, $\beta_2 = \beta_4 = 1/2$, $\beta_3 = -1$



(e) $n = 3$, $\alpha_1 = \alpha_3 = -1/2$, $\alpha_2 = 2$, $\beta_1 = \beta_3 = 3/2$, $\beta_2 = -1$

Figure 2.3: Examples of polygonal regions and the corresponding values of the normalised interior angles α_j and exterior angles β_j. Note that in each case the exterior angles β_j sum to 2.

Example. Map a half-plane to a strip with the vertices corresponding to $z = 0$ and $z = \infty$.



Solution. Here ζ_1 and ζ_2 are both at ∞, with $\beta_1 = \beta_2 = 1$. We choose $x_1 = 0$ and $x_2 = \infty$ and thus the Schwarz–Christoffel formula (2.10) gives

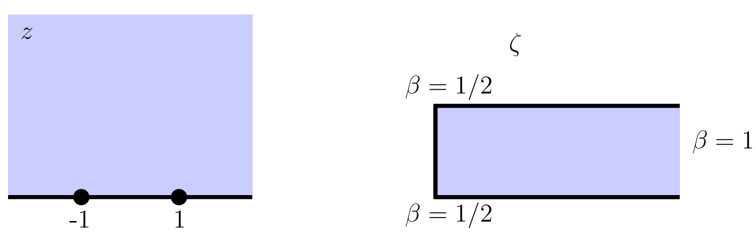
$$\zeta = A + C \int^z \frac{\mathrm{d}t}{t} = A + C \log z. \tag{2.11}$$

If instead we wanted to map general points $z = x_1$ and $z = x_2$ on the real axis to the ends of the strip we would have

$$\zeta = A + C \int^z \frac{\mathrm{d}t}{(t - x_1)(t - x_2)} = A + \tilde{C} \log\left(\frac{z - x_1}{z - x_2}\right). \tag{2.12}$$

The values of A and C set the location, orientation, and width of the strip.

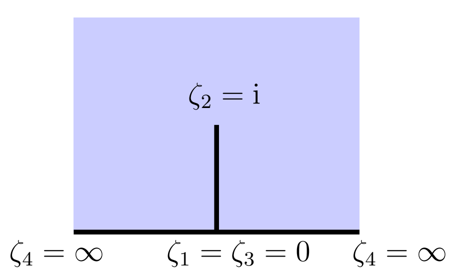
Example. Map a half-plane to a half-strip.



Solution. Here $n = 3$, $\beta_1 = \beta_2 = 1/2$, $\beta_3 = 1$. It is convenient to take $x_1 = -1$, $x_2 = 1$, $x_3 = \infty$, to give

$$\zeta = A + C \int^z \frac{\mathrm{d}t}{\sqrt{t^2 - 1}} = A + C \cosh^{-1} z. \tag{2.13}$$

Example. Map the upper half-plane to the slit domain shown.



Solution. Here we have *four* vertices, with $\beta_1 = 1/2$, $\beta_2 = -1$, $\beta_3 = 1/2$, $\beta_4 = 2$. In general we can only choose three locations for the x_j, but here symmetry allows us to take $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = \infty$. Thus

$$\zeta = A + C \int^z \frac{t}{\sqrt{t^2 - 1}}\,\mathrm{d}t = A + C\sqrt{z^2 - 1}. \tag{2.14}$$

To fix the values of the constants A and C, we must ensure that the vertices end up in the right places:

$$\zeta_1 = \zeta_3 = 0 \quad \Rightarrow \quad \zeta = 0 \text{ when } z = \pm 1 \quad \Rightarrow \quad A = 0, \tag{2.15a}$$

$$\zeta_2 = \mathrm{i} \quad \Rightarrow \quad \zeta = \mathrm{i} \text{ when } z = 0 \quad \Rightarrow \quad C = 1. \tag{2.15b}$$

Thus the required map is

$$\zeta = \sqrt{z^2 - 1}. \tag{2.16}$$

Although this example has 4 vertices, symmetry gives an exact solution.

2.3 Solving Laplace's equation by conformal maps

Models leading to Laplace's equation

Laplace's equation crops up in a wide variety of practically motivated models. Here are three examples.

Example 1: Steady heat flow

Fourier's law of heat conduction states that the heat flux in a homogeneous isotropic medium D of constant thermal conductivity k is

$$\boldsymbol{q} = -k\boldsymbol{\nabla} u, \tag{2.17}$$

where u is the temperature. When the temperature is time-independent, conservation of energy imples that $\boldsymbol{\nabla} \cdot \boldsymbol{q} = 0$, and hence u satisfies Laplace's equation:

$$\nabla^2 u = 0 \qquad \text{in } D. \tag{2.18}$$

At a boundary typically either the temperature u or the heat flux $\boldsymbol{q} \cdot \boldsymbol{n}$ is known. The former case leads to a *Dirichlet* boundary condition, where u is specified on the boundary. The latter case corresponds to the *Neumann* boundary condition, where $\partial u/\partial n$ is specified on the boundary. In particular, $\partial u/\partial n = 0$ at an insulated boundary.

Example 2: Electrostatics

In a steady state, the electric field $\boldsymbol{E}$ satisfies $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$, and may therefore be written in terms of an *electric potential* ϕ such that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi. \tag{2.19}$$

Moreover, in the absence of any space charge, Gauss's Law implies that $\nabla \cdot \boldsymbol{E} = 0$, and therefore ϕ satisfies Laplace's equation

$$\nabla^2 \phi = 0. \tag{2.20}$$

The potential ϕ is the usual voltage we talk about in the context of batteries, mains electricity, lightning etc.

A common and useful boundary condition for ϕ is that it is constant on a good conductor like a metal. Thus a canonical problem is to determine the potential between two perfect conductors each of which is held at a given constant potential (for example in a *capacitor*).

Example 3: Inviscid fluid flow

The simplest model for a fluid is that it is *inviscid*, *incompressible* and *irrotational*. Fortunately this is a remarkably accurate model in many circumstances. In an incompressible fluid, the velocity field $\boldsymbol{u}$ satisfies $\nabla \cdot \boldsymbol{u} = 0$, while an irrotational flow satisfies $\nabla \times \boldsymbol{u} = \boldsymbol{0}$. In two dimensions, with $\boldsymbol{u} = \big(u(x,y), v(x,y)\big)$, the velocity components u and v satisfy the equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \tag{2.21}$$

in an incompressible, irrotational flow. Finally, the pressure p may be found using *Bernoulli's Theorem*, which states that

$$p + \frac{1}{2}\rho|\boldsymbol{u}|^2 = \text{constant} \tag{2.22}$$

in a steady incompressible, irrotational flow, where ρ is the density of the fluid.

From equation (2.21a), we deduce the existence of a potential function $\psi(x,y)$, called the *streamfunction*, such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{2.23}$$

Similarly, equation (2.21b) implies the existence of a *velocity potential* $\phi(x,y)$, such that

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}. \tag{2.24}$$

Thus $\nabla\phi$ is everywhere tangent to the flow, while $\nabla\psi$ is everyhere normal to the velocity. It follows that the contours of ψ are *streamlines* for the flow, i.e. curves everywhere parallel to the velocity. Moreover, the change in the value of ψ on two neighbouring streamlines is equal to the flux of fluid between them. To see this, calculate the net flow across a curve C connecting two streamlines, as shown in Figure 2.4:

$$\begin{aligned}
\text{flux} &= \int_C \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s \\
&= \int_c \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)\cdot(\mathrm{d}y, -\mathrm{d}x) \\
&= \int_c \left(\frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y\right) \\
&= [\psi]_C = \psi_2 - \psi_1,
\end{aligned} \tag{2.25}$$

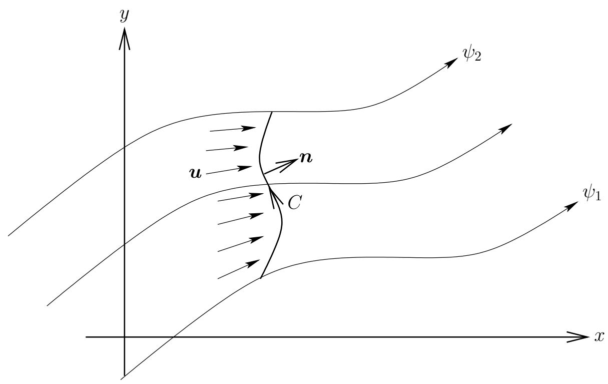


Figure 2.4: Schematic of a curve C joining two streamlines on which $\psi = \psi_1$ and $\psi = \psi_2$.

where ψ_1 and ψ_2 are the constant values of ψ on the two streamlines.

Elimination between (2.23) and (2.24) shows that both ϕ and ψ satisfy Laplace's equation. Alternatively, by combining (2.23) and (2.24), we see that ϕ and ψ satisfy the *Cauchy–Riemann equations*

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.26}$$

Therefore $\phi + \mathrm{i}\psi$ is a holomorphic function of $z = x + \mathrm{i}y$:

$$\phi + \mathrm{i}\psi = w(z), \tag{2.27}$$

where w is called the *complex potential*. The velocity components can be recovered from w using

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - \mathrm{i}v. \tag{2.28}$$

At a fixed impenetrable boundary, the normal component of $\boldsymbol{u}$ must be zero. In terms of the velocity potential and streamfunction, this is equivalent to

$$\frac{\partial \phi}{\partial n} = 0, \qquad \psi = \text{constant}, \tag{2.29}$$

and therefore, in terms of the complex potential,

$$\operatorname{Im} w(z) = \text{constant} \qquad \text{at a fixed impenetrable boundary.} \tag{2.30}$$

Solution by conformal mapping

In all the above examples, we end up having to find a harmonic function u (or ϕ or ψ) in some region D subject to given boundary conditions on ∂D. The general idea is to write u as the real or imaginary part of a holomorphic function $w(z) = u(x, y) + \mathrm{i}v(x, y)$ and then map D onto a simpler domain $f(D)$ by a conformal map

$$\zeta = f(z), \tag{2.31}$$

in the hope that we can more easily find the corresponding function in the ζ-plane, i.e.

$$W(\zeta) = U(\xi, \eta) + \mathrm{i}V(\xi, \eta). \tag{2.32}$$

Because a composition of holomorphic functions is holomorphic, $W(\zeta)$ is holomorphic, and its real and imaginary parts satisfy Laplace's equation in $f(D)$. We then recover the solution in the original domain D by reversing the conformal mapping:

$$w(z) = W\big(f(z)\big). \tag{2.33}$$

Example. Find the temperature u in a domain D exterior to the circles $|z-\mathrm{i}| = 1$, $|z+\mathrm{i}| = 1$ with $u = \pm 1$ on $|z \mp \mathrm{i}| = 1$ and $u \to 0$ at ∞, as depicted in Figure 2.5.

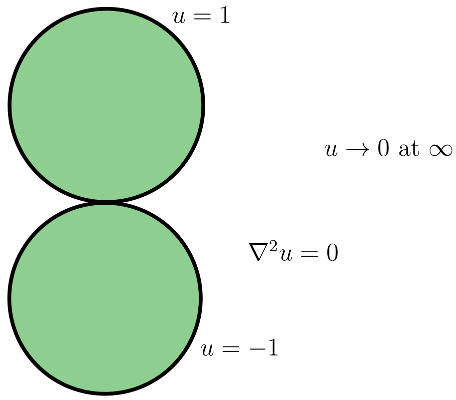


Figure 2.5: Steady heat flow in the region outside two touching circles.

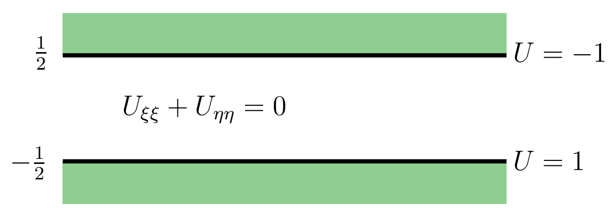


Figure 2.6: The image of the problem from Figure 2.5 under the mapping $\zeta = 1/z$.

Solution. The map $\zeta = 1/z$ takes D onto the strip $-1/2 < \operatorname{Im}\zeta < 1/2$, so the corresponding problem in the ζ-plane is as shown in Figure 2.6. By inspection, the solution is

$$U = -2\eta = 2\,\mathrm{Re}(\mathrm{i}\zeta), \tag{2.34}$$

and hence

$$u = 2\,\mathrm{Re}(\mathrm{i}/z) = \frac{2y}{x^2+y^2}. \tag{2.35}$$

Note that u is bounded in all of D, since $|y| \lesssim x^2/2$ as $(x, y) \to (0, 0)$.

Example. Calculate the complex potential for flow past the unit circle with uniform velocity $(U_\infty, 0)$ at ∞.

Solution. The complex potential is $w = \phi + \mathrm{i}\psi$ and we can take $\psi = 0$ on the x-axis and unit circle. Also $w \sim U_\infty z$ at ∞. Under the Jowkowski map

$$\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{2.36}$$

the exterior of the unit circle maps to the ζ-plane cut from -1 to 1, as shown in Figure 2.7.

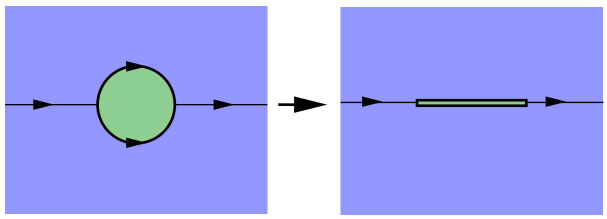


Figure 2.7: Uniform flow past a circle transformed by the Joukowski map.

In the ζ-plane we need a function $W(\zeta)$ which is real on the ξ axis and at ∞ looks like $U_\infty z \sim 2U_\infty\zeta$. The solution is just

$$W(\zeta) = 2U_\infty\zeta \qquad \Rightarrow \qquad w(z) = U_\infty\left(z + \frac{1}{z}\right). \tag{2.37}$$

Example. Find the complex potential for flow over a step of height 1, from $y = 1$, $x < 0$ to $y = 0$, $x > 0$, with velocity $(U_\infty, 0)$ at ∞.

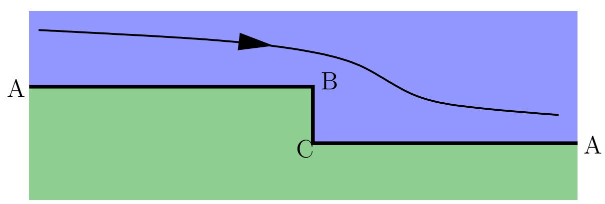


Figure 2.8: Flow over a step.

Solution. The flow is depicted in Figure 2.8. We map the half-plane $\mathrm{Im}\, Z > 0$ onto D by Schwarz–Christoffel (using Z not ζ because the roles are reversed). The exterior angles at the marked vertices are given by

$$\beta_A = 2, \qquad \beta_B = -\frac{1}{2}, \qquad \beta_B = \frac{1}{2}. \tag{2.38}$$

Since there are just three vertices, we can choose to map

$$Z = -1 \text{ to } B, \qquad Z = +1 \text{ to } C, \qquad Z = \infty \text{ to } A. \tag{2.39}$$

Then the Schwarz–Christoffel formula (2.10) gives

$$z = A + C\int^Z \left(\frac{t+1}{t-1}\right)^{1/2} \mathrm{d}t = A + C\left((Z^2-1)^{1/2} + \cosh^{-1} Z\right). \tag{2.40}$$

From the conditions (2.39) we find $A = 0$ and $C = 1/\pi$, so the required mapping function is given by

$$z = \frac{1}{\pi}\left((Z^2-1)^{1/2} + \cosh^{-1} Z\right). \tag{2.41}$$

At infinity $z \sim Z/\pi$, so the specified uniform flow at infinity implies that

$$w(z) \sim U_\infty z \text{ at } \infty \qquad \Rightarrow \qquad W(Z) = w(z(Z)) \sim \frac{U_\infty Z}{\pi} \text{ at } \infty. \tag{2.42}$$

Thus the flow in the Z plane is given by

$$W(Z) = \frac{U_\infty Z}{\pi}, \tag{2.43}$$

so that $w(z)$ is given implicitly by

$$z = \frac{1}{\pi}\left(\left(\frac{\pi^2 w^2}{U_\infty^2} - 1\right)^{1/2} + \cosh^{-1}\frac{\pi w}{U_\infty}\right). \tag{2.44}$$

Note that the velocity components may be found using

$$\begin{aligned} u - \mathrm{i}v &= \frac{\mathrm{d}w}{\mathrm{d}z} \\ &= \frac{\mathrm{d}W/\mathrm{d}Z}{\mathrm{d}z/\mathrm{d}Z} \\ &= U_\infty\left(\frac{Z-1}{Z+1}\right)^{1/2}. \end{aligned} \tag{2.45}$$

Therefore the velocity is zero at C ($Z = 1$) and infinite at B ($Z = -1$). In general, at a corner with interior angle γ, the complex potential locally is of the form $w \sim \text{constant} \times z^{\pi/\gamma}$ and therefore $u - \mathrm{i}v \sim \text{constant} \times z^{\pi/\gamma - 1}$, which implies that:

The velocity is zero *at a corner with interior angle* $< \pi$
and infinite *at a corner with interior angle* $> \pi$. (2.46)

Example. A lightning conductor is modelled by the boundary-value problem illustrated in Figure 2.9. The potential u is equal to zero at $x = 0$ and satisfies Laplace's equation in the half-space $x > 0$, except on the line $y = 0$, $x > 1$, where $u = 1$. We also require u to be bounded at infinity.

Solution. The domain D is the image of the strip $0 < X < \pi/2$, $-\infty < Y < \infty$ under the map $z = \sin Z$. In the Z-plane we have $\nabla^2 U = 0$ with $U = 0$ at $X = 0$ and $U = 1$ at $X = \pi/2$, as shown in Figure 2.10. The solution in the Z-plane is $U = \operatorname{Re} W = \operatorname{Re}(2Z/\pi)$, and therefore

$$u = \frac{2}{\pi}\operatorname{Re}\left(\sin^{-1} z\right). \tag{2.47}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{2}{\pi}\sin^{-1} z\right) = \frac{2}{\pi}\frac{1}{\sqrt{1-z^2}}, \tag{2.48}$$

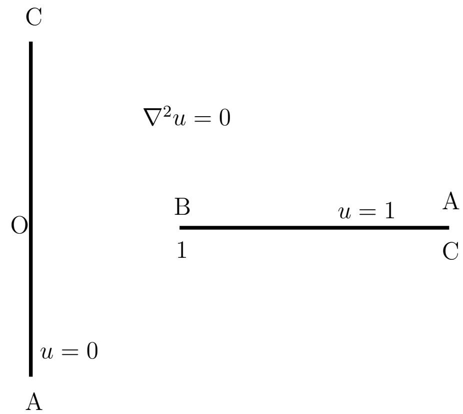


Figure 2.9: A model for a lightning conductor.

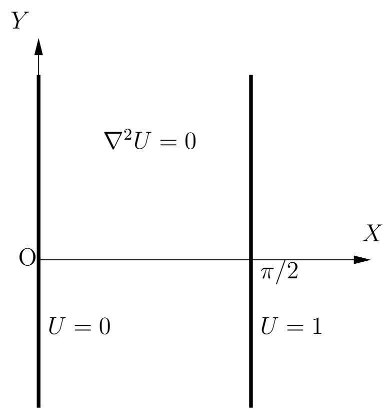


Figure 2.10: The problem from Figure 2.9 transformed by the map $Z = \sin^{-1} z$.

and it follows that $|\boldsymbol{\nabla} u| \to \infty$ as $z \to 1$, i.e. at the tip of the spike.

This example could also have been solved by using a Schwarz–Christoffel mapping to map the upper half Z-plane to D. The vertices marked A, B, C in Figure 2.9 have the exterior angles

$$\beta_A = \frac{3}{2}, \qquad \beta_B = -1, \qquad \beta_C = \frac{3}{2}. \tag{2.49}$$

We can choose to map

$$Z = -1 \text{ to } A, \qquad Z = 0 \text{ to } B, \qquad Z = 1 \text{ to } C, \tag{2.50}$$

and, because of symmetry, we are also free to map $Z = \infty$ to $z = 0$. Then the Schwarz–Christoffel formula gives

$$z = \frac{1}{\sqrt{1 - Z^2}}. \tag{2.51}$$

The problem in the Z plane is shown in Figure 2.11. The solution bounded at $Z = \pm 1$ is simply

$$U = \frac{1}{\pi}\big(\arg(Z+1) - \arg(Z-1)\big) = \operatorname{Im}\left[\frac{1}{\pi}\log\left(\frac{Z-1}{Z+1}\right)\right], \tag{2.52}$$

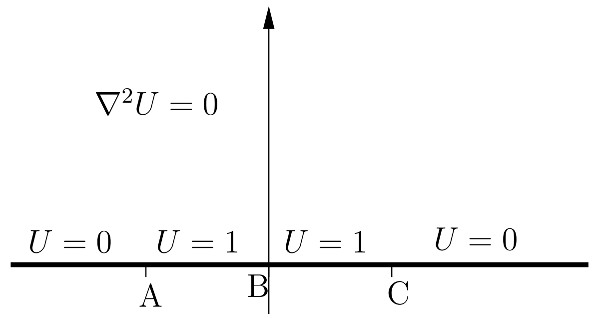


Figure 2.11: The problem from Figure 2.9 transformed by the map $z = 1/\sqrt{1 - Z^2}$.

and by inverting the conformal map we find

$$u = \frac{1}{\pi} \operatorname{Im} \left[\log \left(\frac{\sqrt{z^2 - 1} - z}{\sqrt{z^2 - 1} + z} \right) \right], \tag{2.53}$$

which is equivalent to (2.47).