5 Complex Fourier Transforms

5.1 Introduction

Below we summarise the main properties of the standard Fourier transform. One of the main restrictions on the Fourier transform is that it only applies to functions which decay sufficiently rapidly at infinity. Here we will show how this restriction may be lifted by extending the concept of the Fourier transform into the complex plane. We will then show how this generalised Fourier transform may be used to solve some linear differential equations.

Basic properties of the Fourier transform

Given an integrable function $f(x)$, we define the **Fourier transform** of f by

$$\mathcal{F}[f(x)] = \bar{f}(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x. \tag{5.1}$$

Given the transform $\bar{f}(k)$, then we can recover $f(x)$ using the **inverse Fourier transform**:

$$\frac{1}{2}\big(f(x_-)+f(x_+)\big) = \mathcal{F}^{-1}\left[\bar{f}(k)\right] = \frac{1}{2\pi}⨍_{-\infty}^{\infty} \bar{f}(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k, \tag{5.2}$$

where the prinicipal value integral is defined by

$$⨍_{-\infty}^{\infty} = \lim_{R\to\infty}\int_{-R}^{R}. \tag{5.3}$$

If $f(x)$ is continuous, then the left-hand side of equation (5.2) is just $f(x)$. Moreover, the integrand on the right-hand side of equation (5.2) is integrable and the dash may be removed from the integral sign.

We will just list without proof some useful properties of the Fourier transform, assuming the required integrability where necessary. First, the **Fourier transform of a derivative** is given by

$$\mathcal{F}\left[f'(x)\right] = -\mathrm{i}k\bar{f}(k), \tag{5.4}$$

which is easily shown using integration by parts. Differentiation under the integral sign in (5.1) gives the **derivative of a Fourier transform**, namely

$$\frac{\mathrm{d}\bar{f}(k)}{\mathrm{d}k} = \mathrm{i}\mathcal{F}[xf(x)]. \tag{5.5}$$

Equation (5.4) shows that $\mathcal{F}$ will turn a linear differential equation for $f(x)$ into a linear algebraic equation for $\bar{f}(k)$. Once we have solved for $\bar{f}(k)$, in principle we can recover $f(x)$ from the inversion formula (5.2), typically using contour integration in the complex k-plane. In practice this is rarely necessary: one can look up very many Fourier transforms that often arise, and the following is also useful. The **inverse Fourier transform of a product** is given by

$$\mathcal{F}^{-1}\left[\bar{f}(k)\bar{g}(k)\right] = (f \star g)(x), \tag{5.6}$$

where $f \star g$ denotes the **convolution** of f and g, defined by

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(s)g(x-s)\,\mathrm{d}s. \tag{5.7}$$

Example: solving Laplace's equation by Fourier transform. Find $u(x, y)$ that satisfies Laplace's equation in the half-space $y > 0$, with the boundary condition $u(x, 0) = u_0(x)$ on $y = 0$, and such that $u(x, y)$ is bounded as $y \to +\infty$.

Solution. We take the Fourier transform in x, i.e.

$$\bar{u}(k, y) = \int_{-\infty}^{\infty} u(x, y)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x. \tag{5.8}$$

The whole problem is transformed as follows:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \bar{u}}{\partial y^2} - k^2\bar{u} = 0, \tag{5.9a}$$

$$u = u_0(x) \text{ at } y = 0 \quad \Rightarrow \quad \bar{u} = \bar{u}_0(k) \text{ at } y = 0, \tag{5.9b}$$

$$u \text{ bounded as } y \to \infty \quad \Rightarrow \quad \bar{u} \text{ bounded as } y \to \infty, \tag{5.9c}$$

where $\bar{u}_0(k)$ is the Fourier transform of $u_0(x)$.

The general solution of (5.9a) is

$$\bar{u}(k, y) = A(k)\mathrm{e}^{ky} + B(k)\mathrm{e}^{-ky}, \tag{5.10}$$

where A and B are arbitrary integration functions. We need to make sure that $\bar{u}$ is bounded as $y \to \infty$. Which of the two exponentials in (5.11) should be kept depends on k: if $k > 0$ then A must be zero, while if $k < 0$ then B must be zero. Both of these cases may be encompassed by setting

$$\bar{u}(k, y) = C(k)\mathrm{e}^{-|k|y}, \tag{5.11}$$

so that the decaying exponential is selected regardless of the sign of k. Finally we apply the boundary condition (5.9b) to get

$$\bar{u}(k, y) = \bar{u}_0(k)\mathrm{e}^{-|k|y}. \tag{5.12}$$

This shows how the Fourier transform converts a PDE for u to an ODE for $\bar{u}$ which is then easily solved. However, it remains to invert (5.12) to find u. Here we can use the convolution theorem (5.6) to get

$$u(x, y) = u_0(x) \star g(x, y) = \int_{-\infty}^{\infty} u_0(s)g(x-s, y)\,\mathrm{d}s, \tag{5.13}$$

where

$$g(x, y) = \mathcal{F}^{-1}\left[\mathrm{e}^{-|k|y}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-|k|y-\mathrm{i}kx}\,\mathrm{d}k = \frac{y}{\pi\left(x^2+y^2\right)}. \tag{5.14}$$

Thus we obtain the general solution

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u_0(s)}{(x-s)^2+y^2}\,\mathrm{d}s. \tag{5.15}$$

5.2 Complex Fourier transform

Our aim is to generalise the Fourier transform to a wider class of functions by allowing k to take complex values. For simplicity we will assume that $f(x)$ is continuous, although this assumption can be relaxed relatively easily. We also assume that f grows at most exponentially at infinity, that is $f(x) = O\left(\mathrm{e}^{c|x|}\right)$ as $x \to \pm\infty$ for some constant $c > 0$; this rules out $f(x) = \mathrm{e}^{x^2}$, for example.

To investigate the convergence of the Fourier integral (5.1) as $x \to \pm\infty$, we split up $f(x)$ by writing

$$f(x) = f_+(x) + f_-(x), \tag{5.16}$$

where

$$f_+(x) = 0 \quad \text{for } x < 0, \qquad\qquad f_-(x) = 0 \quad \text{for } x > 0. \tag{5.17}$$

From the definition of the Fourier transform,

$$\bar{f}_+(k) = \int_0^\infty f_+(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \int_0^\infty f_+(x)\mathrm{e}^{\mathrm{i}\,\mathrm{Re}(k)x}\mathrm{e}^{-\,\mathrm{Im}(k)x}\,\mathrm{d}x, \tag{5.18}$$

we see that

$$\left|\bar{f}_+(k)\right| \leq \int_0^\infty |f(x)|\mathrm{e}^{-\,\mathrm{Im}(k)\,x}\,\mathrm{d}x, \tag{5.19}$$

and the integral converges provided $\mathrm{Im}\,k > c$. Thus $\bar{f}_+(k)$ exists and is holomorphic for $\mathrm{Im}\,k > c$, since its derivative

$$\frac{\mathrm{d}\bar{f}_+}{\mathrm{d}k} = \mathrm{i}\mathcal{F}\left[xf_+(x)\right] = \mathrm{i}\int_0^\infty xf_+(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \tag{5.20}$$

likewise exists for $\mathrm{Im}\,k > c$.

Next we need to extend the Fourier inversion theorem to recover $f_+(x)$ from $\bar{f}_+(k)$. To this end, let $F_+(x) = \mathrm{e}^{-\alpha x}f_+(x)$, where $\alpha > c$, so that $\bar{F}_+(k) = \bar{f}_+(k + \mathrm{i}\alpha)$ exists and is holomorphic for $\mathrm{Im}\,k > c - \alpha$, in particular for $k \in \mathbb{R}$, since $\alpha > c$. Thus we can apply the Fourier Inversion Theorem (5.2), which gives

$$F_+(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \bar{F}_+(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{5.21a}$$

$$\Rightarrow \quad \mathrm{e}^{-\alpha x}f_+(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \bar{f}_+(k+\mathrm{i}\alpha)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k \tag{5.21b}$$

$$\Rightarrow \quad f_+(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \bar{f}_+(k+\mathrm{i}\alpha)\mathrm{e}^{-\mathrm{i}(k+i\alpha)x}\mathrm{d}k. \tag{5.21c}$$

The final integral corresponds to integration along a horizontal contour in the complex k-plane, i.e.

$$f_+(x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \bar{f}_+(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k, \tag{5.22}$$

where the integration contour is as shown in Figure 5.1.

Suppose $\bar{f}_+(k)$ can be continued below $\mathrm{Im}\,k = c$, so that it is holomorphic in some region $\Omega_+ \supset \{k : \mathrm{Im}\,k > c\}$ except for singularities at $k = a_1, a_2, \cdots$. By the deformation theorem,

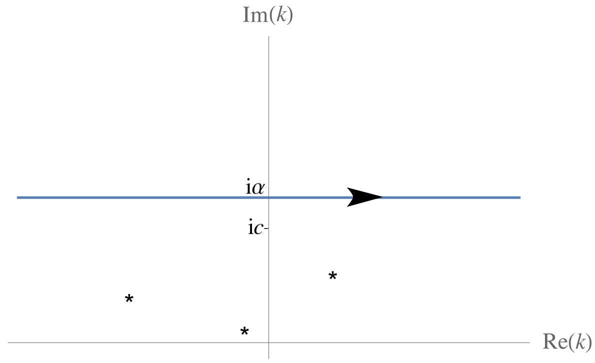

Figure 5.1: Inversion contour for complex Fourier transform $\bar{f}_+(k)$, where $f_+(x) = O\left(\mathrm{e}^{cx}\right)$ as $x \to \infty$.

the inversion contour $\Gamma_+ = \{x + i\alpha : -\infty < x < \infty\}$ may be deformed into Ω_+ provided it passes *above* all the singularities of $\bar{f}_+$, as shown in Figure 5.2(a). Since the singularities of $\bar{f}_+(k)$ are below the inversion contour, for $x < 0$ we can close the inversion contour at $+\mathrm{i}\infty$, as shown in Figure 5.2(b). This gives this expected result that $f_+(x) = 0$ for $x < 0$. For $x > 0$, we would need to close the contour in $\operatorname{Im} k < 0$, picking up the contributions from the singularities in $\bar{f}_+(k)$ and giving a nonzero value of $f_+(x)$.

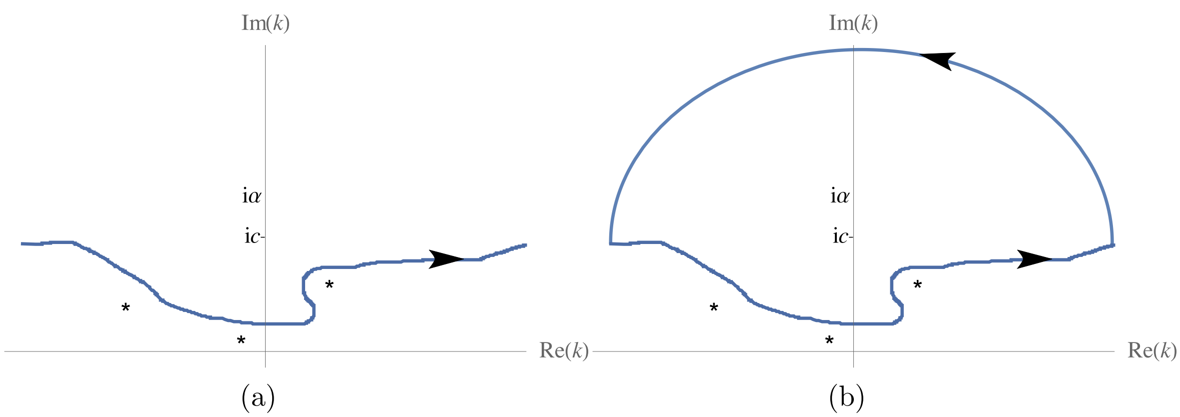

Figure 5.2: (a) An inversion contour that passes above the singularities ($\star$) in $\bar{f}_+(k)$. (b) Closing the contour at infinity when $x < 0$.

The same procedure works for $f_-(x)$ with everything upside down: $\bar{f}_-(k)$ exists and is holomorphic for $\operatorname{Im} k < -c$, while an application of the Fourier Inversion Theorem to $F_-(x) = \mathrm{e}^{\beta x} f_-(x)$ gives

$$f_-(x) = \frac{1}{2\pi} \int_{-\infty - \mathrm{i}\beta}^{\infty - \mathrm{i}\beta} \bar{f}_-(k) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}k \tag{5.23}$$

provided $-\beta < -c$. Suppose $\bar{f}_-(k)$ can be continued above $\operatorname{Im} k = -c$, so that it is holomorphic in some region $\Omega_- \supset \{k : \operatorname{Im} k < -c\}$ except for singularities at $k = b_1, b_2, \cdots$. By the deformation theorem, the inversion contour $\Gamma_- = \{x - i\beta : -\infty < x < \infty\}$ may be deformed into Ω_- provided it passes *underneath* the singularities b_j of $\bar{f}_-$.

If there is a non-empty *overlap region* $\Omega = \Omega_+ \cap \Omega_- \setminus (\{a_j\} \cup \{b_j\})$, then the Fourier

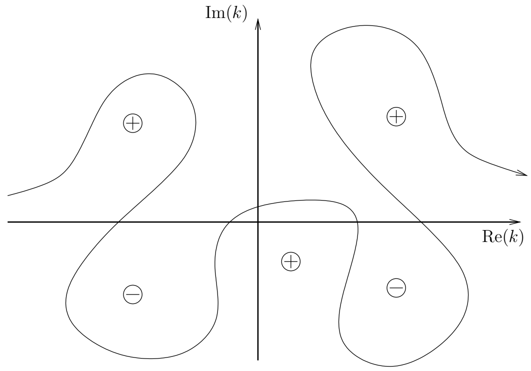

Figure 5.3: An inversion contour that passes above the singularities ($\oplus$) in $\bar{f}_+(k)$ and below the singularities ($\ominus$) in $\bar{f}_-(k)$.

transform of f is defined by

$$\bar{f}(k) = \bar{f}_+(k) + \bar{f}_-(k) \tag{5.24}$$

for $k \in \Omega$. Moreover, if Γ_+ and Γ_- can be deformed into the same contour $\Gamma \subset \Omega$, with Γ *above* the singularities of $\bar{f}_+(k)$ and *below* the singularities of $\bar{f}_-(k)$, as illustrated in figure 5.3, then

$$f(x) = \frac{1}{2\pi}\int_\Gamma \bar{f}(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{5.25}$$

Note that we need Γ to extend from $\operatorname{Re} k = -\infty$ to $\operatorname{Re} k = +\infty$ and $\{a_j\} \cap \{b_j\}$ to be empty, i.e. no singularities are shared by $\bar{f}_+(k)$ and $\bar{f}_-(k)$.

Example. Fourier transform of the Heaviside function The Heaviside function

$$\mathrm{H}(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0, \end{cases} \tag{5.26}$$

has Fourier transform

$$\bar{\mathrm{H}}(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \left[\frac{\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}\right]_0^\infty = \frac{\mathrm{i}}{k} \tag{5.27}$$

provided $\operatorname{Im} k > 0$. We can analytically continue $\bar{\mathrm{H}}(k)$ into $\mathbb{C}\setminus\{0\}$ because i/k is holomorphic except for a simple pole at $k = 0$. When we invert, the inversion contour must pass *above* the pole at $k = 0$:

$$\mathrm{H}(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \frac{\mathrm{e}^{-\mathrm{i}kx}}{k}\,\mathrm{d}k, \qquad \text{where } \alpha > 0. \tag{5.28}$$

If $x < 0$ we can close the contour in the upper half plane to find by Cauchy's Theorem that $\mathrm{H}(x) = 0$ for $x < 0$. For $x > 0$ we need to close the contour in the lower half plane, and we pick up a residue contribution from the pole at the origin (note the minus sign since we are integrating clockwise round the pole) to find

$$\mathrm{H}(x) = -2\pi\mathrm{i} \times \left(\frac{\mathrm{i}}{2\pi}\right) = 1 \qquad \text{for } x > 0. \tag{5.29}$$

The inversion contours for $x > 0$ and $x < 0$ are illustrated in Figure 5.4.

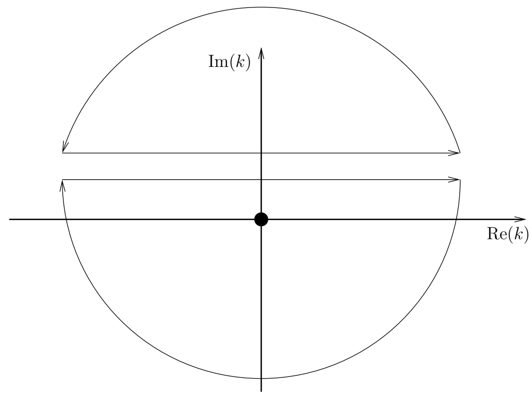

Figure 5.4: The inversion contours for the Heaviside function. For $x < 0$ we close in the upper half-plane; for $x > 0$ we close in the lower half-plane, picking up the residue from the pole at $k = 0$.

The Laplace transform

If we set $k = \mathrm{i}p$ (with p complex) and $\bar{f}_+(k) = \hat{f}_+(p)$, then the Fourier transform (5.18) takes the form

$$\bar{f}_+(k) = \hat{f}_+(p) = \int_0^\infty f_+(x)e^{-px}\,\mathrm{d}x, \tag{5.30}$$

while the inversion formula (5.22) is transformed to

$$f_+(x) = \frac{1}{2\pi\mathrm{i}}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \hat{f}_+(p)e^{px}\,\mathrm{d}p. \tag{5.31}$$

Equation (5.30) defines the *Laplace transform* of a function $f_+(x) : [0,\infty) \mapsto \mathbb{R}$ such that $f_+(x) = O\left(\mathrm{e}^{cx}\right)$ as $x \to \infty$. Equation (5.31) is the Laplace transform inversion formula, where now α must be sufficiently large that the inversion contour lies to the *right* of any singularities of $\hat{f}(p)$, as illustrated in Figure 5.5. So we see that the Laplace transform is just a special case of the Fourier Transform if we allow complex values of k.

5.3 Complex Fourier transform with multifunctions

Example. Find a function $u(x,y)$ which satisfies Laplace's equation in the upper half-plane $y > 0$, which is bounded as $x^2 + y^2 \to \infty$ and which is equal to the Heaviside function on $y = 0$, that is $u(x,0) = \mathrm{H}(x)$.

Solution. As we will see, it is straightforward to spot the appropriate harmonic function $u(x,y)$, but the aim of this example is to illustrate the solution procedure using the complex Fourier transform. The Fourier transform of $u(x,y)$ satisfies the problem

$$\frac{\partial^2\bar{u}}{\partial y^2} - k^2\bar{u} = 0 \qquad \text{in } y > 0, \tag{5.32a}$$

$$\bar{u} = \bar{\mathrm{H}}(k) = \frac{\mathrm{i}}{k} \qquad \text{at } y = 0, \tag{5.32b}$$

$$|\bar{u}| < \infty \qquad \text{as } y \to \infty. \tag{5.32c}$$

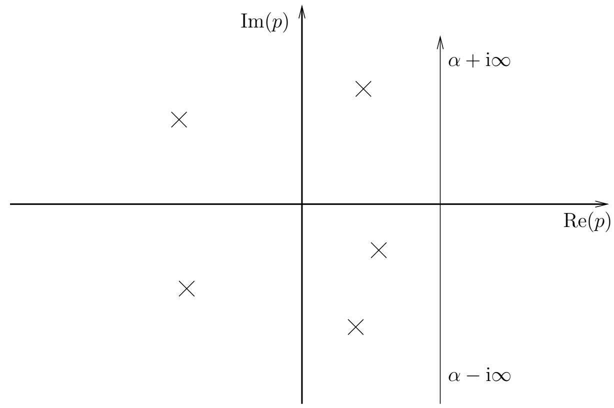

Figure 5.5: The Laplace transform inversion contour passes to the right of all singularities in $\hat{f}_+(p)$.

We recall that the Fourier transform $\bar{\mathrm{H}}(k)$ is defined for $\operatorname{Im} k > 0$ and can be analytically extended onto $\mathbb{C} \setminus \{0\}$.

As we found in equation (5.11), the general solution to (5.32a) which does not grow exponentially as $y \to \infty$ is $C(k)\mathrm{e}^{-|k|y}$ *if* k *is real.* When $k \in \mathbb{C}$, this approach does not work, since $|k|$ is not a holomorphic function of k so that none of the tools of complex analysis (e.g. Cauchy's theorems) can be used for the Fourier inversion.

To avoid this difficulty, we approximate k by a function that *is* holomorphic in a neighbourhood of the real k-axis, namely

$$|k| \approx |k|_\epsilon = \left(k^2 + \epsilon^2\right)^{1/2}, \tag{5.33}$$

where $0 < \epsilon \ll 1$. The branch of this multifunction is chosen such that the branch cuts are along the imaginary k-axis on the intervals $(-\mathrm{i}\infty, -\mathrm{i}\epsilon]$ and $[\mathrm{i}\epsilon, \mathrm{i}\infty)$, as shown in Figure 5.6, and $\left(k^2 + \epsilon^2\right)^{1/2} = \sqrt{k^2 + \epsilon^2} > 0$ when k is real. Then $|k|_\epsilon$ defines a function that is holomorphic on the cut complex plane, and $|k|_\epsilon \to |k|$ as $\epsilon \to 0$ when k is real.

Using this approximation, we write the solution for $\bar{u}$ as

$$\bar{u}(k, y) = \frac{\mathrm{i}}{k}\, \mathrm{e}^{-y\left(k^2+\epsilon^2\right)^{1/2}}. \tag{5.34}$$

Note that the solution (5.34) corresponds to solving the modified Helmholtz equation

$$\nabla^2 u = \epsilon^2 u \tag{5.35}$$

instead of Laplace's equation. The idea now is to invert (using contour integration) to find $u(x, y)$, and then let $\epsilon \to 0$. The inversion formula (5.22) gives

$$u(x, y) = \frac{\mathrm{i}}{2\pi} \int_\Gamma \mathrm{e}^{-y\left(k^2+\epsilon^2\right)^{1/2} - \mathrm{i}kx}\, \frac{\mathrm{d}k}{k}, \tag{5.36}$$

where the inversion contour Γ passes between the pole at $k = 0$ and the branch point at $k = \epsilon$, as shown in Figure 5.6. Now the contour used to evaluate $u(x, y)$ depends on the sign of x.

If x is negative, then to ensure that the exponential in (5.36) decays at infinity, we close the integration contour in the upper half-plane, as illustrated in Figure 5.7(a). This results

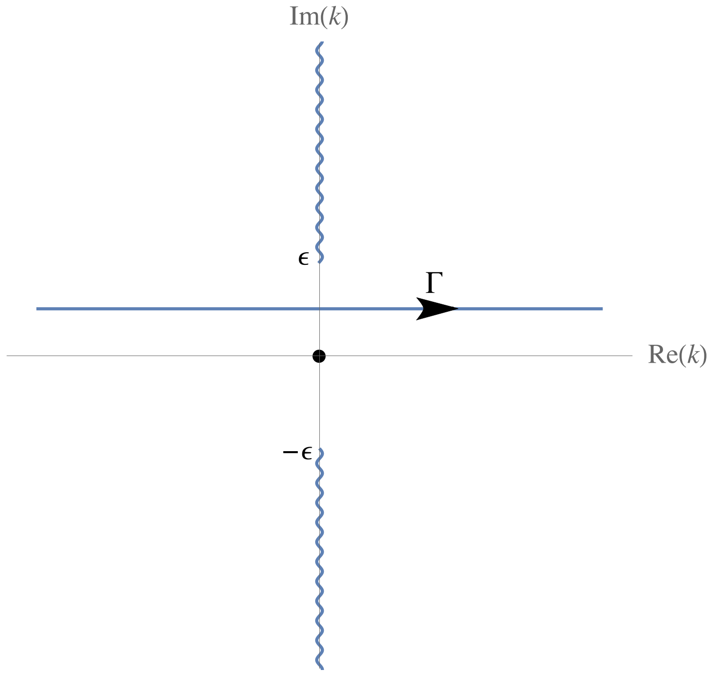

Figure 5.6: Inversion contour for (5.36) passing between the pole at $k = 0$ and the branch point at $k = \epsilon$. The branch cuts for the multifunction $\left(k^2 + \epsilon^2\right)^{1/2}$ are on the intervals $(-\mathrm{i}\infty, -\mathrm{i}\epsilon]$ and $[\mathrm{i}\epsilon, \mathrm{i}\infty)$ along the imaginary k-axis.

in two integrals along either side of the branch cut, which may be parameterised using $k = \mathrm{i}t$ with $t \in (\epsilon, \infty)$ and $\left(k^2 + \epsilon^2\right)^{1/2} = \pm\mathrm{i}\sqrt{t^2 - \epsilon^2}$ on $k = \mathrm{i}t \pm 0$. The resulting integrals may then be combined to give

$$u(x, y) = \frac{1}{\pi} \int_{\epsilon}^{\infty} \sin\left(y\sqrt{t^2 - \epsilon^2}\right) \mathrm{e}^{tx} \frac{\mathrm{d}t}{t} \qquad \text{for } x < 0. \tag{5.37}$$

Letting $\epsilon \to 0$, we find

$$u(x, y) = \frac{1}{\pi} \int_0^{\infty} \sin(yt)\, \mathrm{e}^{tx} \frac{\mathrm{d}t}{t} = \frac{1}{\pi} \int_0^{\infty} \sin(r \sin\alpha)\, \mathrm{e}^{-r\cos\alpha} \frac{\mathrm{d}r}{r} \qquad \text{for } x < 0, \tag{5.38}$$

where $\sin\alpha = y/\sqrt{x^2 + y^2}$ and $\cos\alpha = -x/\sqrt{x^2 + y^2}$. By applying Cauchy's Theorem to the function e^{-z}/z on the closed contour sketched in Figure 5.8, we obtain

$$\lim_{\epsilon \to 0} \left\{ \int_{\epsilon}^{\infty} \frac{\mathrm{e}^{-x}\,\mathrm{d}x}{x} - \int_{\epsilon}^{\infty} \mathrm{e}^{-r\cos\alpha} \left[\cos(r\sin\alpha) - \mathrm{i}\sin(r\sin\alpha)\right] \frac{\mathrm{d}r}{r} - \mathrm{i}\alpha \right\} = 0, \tag{5.39}$$

and the imaginary part gives

$$\int_0^{\infty} \mathrm{e}^{-r\cos\alpha} \sin(r\sin\alpha) \frac{\mathrm{d}r}{r} = \alpha. \tag{5.40}$$

Therefore the integral in (5.38) may be evaluated to give the solution

$$u(x, y) = \frac{1}{\pi} \tan^{-1}\left(\frac{y}{-x}\right) \qquad \text{for } x < 0. \tag{5.41}$$

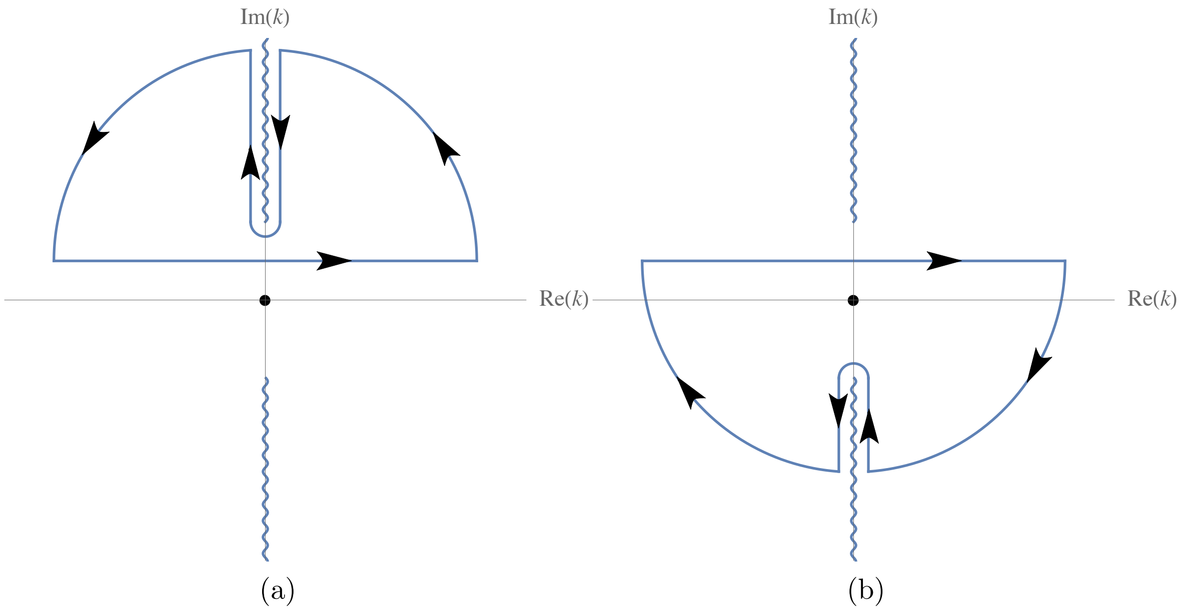

Figure 5.7: Inversion contours for (5.36). (a) If $x < 0$ the contour is closed in the upper half-plane. (b) If $x > 0$ the contour is closed in the lower half-plane, picking up the residue from the pole at $k = 0$.

If x is positive, then we evaluate (5.36) by closing the integration contour in the lower half-plane, as illustrated in Figure 5.7(b). This time we get two integrals along either side of the branch cut on the negative imaginary k-axis, as well as the residue from the pole at $k = 0$, resulting in

$$u(x,y) = 1 - \frac{1}{\pi}\int_0^\infty \sin(yt)\,\mathrm{e}^{-xt}\,\frac{\mathrm{d}t}{t} = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right) \qquad \text{for } x > 0. \tag{5.42}$$

Finally, combining the solutions (5.41) and (5.42) in $x < 0$ and $x > 0$, we see that the solution is simply

$$u(x,y) = 1 - \frac{\theta}{\pi}, \tag{5.43}$$

where θ is the usual polar angle. As pointed out earlier, we might have spotted this simple solution of Laplace's equation straight away, either by using polar coordinates or by writing it as $1 - 1/\pi\,\mathrm{Im}(\log z)$.

5.4 Integral solutions of differential equations

Now we show how a class of linear ordinary differential equations may be solved by a generalised complex Fourier transform. We illustrate the general method using the simple first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy. \tag{5.44}$$

Our aim is to represent the solution as a generalised Fourier integral

$$y(x) = \int_\Gamma g(\zeta)\mathrm{e}^{x\zeta}\,\mathrm{d}\zeta, \tag{5.45}$$

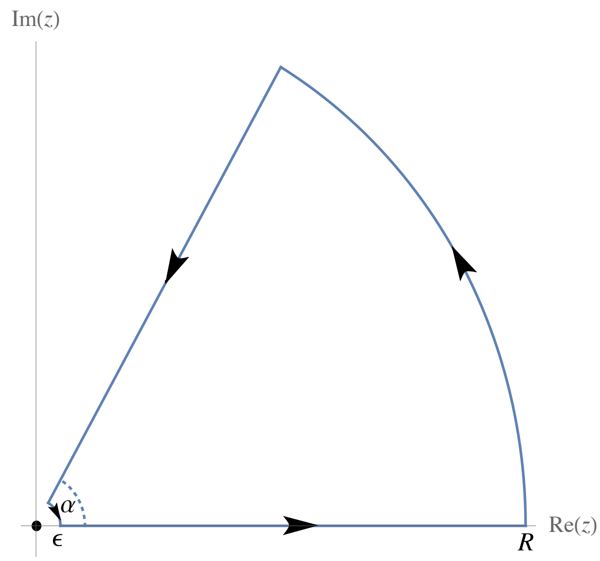

Figure 5.8: Closed contour for the integral of e^{-z}/z.

where both the function $g(\zeta)$ and the integration contour Γ are to be determined.

By differentiating under the integral sign and then integrating by parts, we get

$$\begin{aligned} 0 &= \frac{\mathrm{d}y}{\mathrm{d}x} - xy \\ &= \int_\Gamma \big(\zeta g(\zeta) - x g(\zeta)\big)\mathrm{e}^{x\zeta}\,\mathrm{d}\zeta \\ &= \Big[-g(\zeta)\mathrm{e}^{x\zeta}\Big]_\Gamma + \int_\Gamma \big(\zeta g(\zeta) + g'(\zeta)\big)\mathrm{e}^{x\zeta}\,\mathrm{d}\zeta. \end{aligned} \tag{5.46}$$

We require this equation to be satisfied for all x, and also the integration contour Γ to be independent of x. It follows that we will have a solution to the differential equation (5.44) only if

$$g'(\zeta) = -\zeta g(\zeta) \tag{5.47}$$

and the change in $g(\zeta)\mathrm{e}^{x\zeta}$ around Γ is zero. Integration of (5.47) gives

$$g(\zeta) = C\mathrm{e}^{-\zeta^2/2}, \tag{5.48}$$

for some constant C, and hence

$$y(x) = C\int_\Gamma \mathrm{e}^{x\zeta-\zeta^2/2}\,\mathrm{d}\zeta. \tag{5.49}$$

For the integral to exist, we need the integrand to decay as $|\zeta| \to \infty$, which is true provided $\mathrm{Re}\left[\zeta^2\right] > 0$, which occurs in two sectors $-\pi/4 < \arg\zeta < \pi/4$ and $3\pi/4 < \arg\zeta < 5\pi/4$. Thus the contour Γ must start and end in one of these "valleys". If Γ starts and ends in the same valley the integral (5.49) evaluates to zero by Cauchy's Theorem. Thus there is just one independent solution, corresponding to a contour Γ which starts in one valley and ends in the other, as shown in Figure 5.9. For example, we can simply take Γ along the real axis to

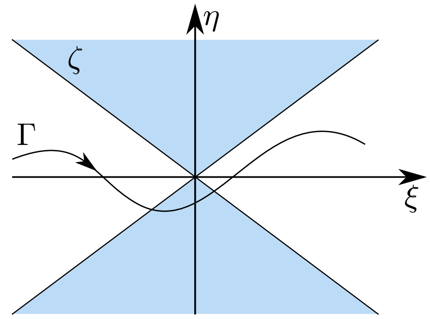

Figure 5.9: A possible integration contour Γ for equation (5.49).

obtain

$$y(x) = C\int_{-\infty}^{\infty} \mathrm{e}^{x\xi-\zeta^2/2}\,\mathrm{d}\xi = Ce^{x^2/2}\int_{-\infty}^{\infty} \mathrm{e}^{-(x-\xi)^2/2}\,\mathrm{d}\xi = C\sqrt{2\pi}\,\mathrm{e}^{x^2/2}. \qquad (5.50)$$

Of course we could easily have found the solution (5.50) directly from the first-order ODE (5.44). The method becomes more useful when applied to higher-order ODEs which are not directly solvable in terms of elementary functions. For example, consider **Airy's equation**

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + xy = 0. \qquad (5.51)$$

We again seek a solution for $y(x)$ in the form of the generalised Fourier integral (5.45), and again differentiate under the integral and integrate by parts, this time arriving at

$$\Big[g(\zeta)\mathrm{e}^{x\zeta}\Big]_\Gamma + \int_\Gamma \big(\zeta^2 g(\zeta) - g'(\zeta)\big)\,\mathrm{e}^{x\zeta}\,\mathrm{d}\eta = 0. \qquad (5.52)$$

Hence, Airy's equation (5.51) is satisfied only if

$$g'(\zeta) = \zeta^2 g(\zeta) \qquad \text{and} \qquad \Big[g(\zeta)\mathrm{e}^{x\zeta}\Big]_\Gamma = 0. \qquad (5.53)$$

Thus, $g(\zeta) = Ce^{\zeta^3/3}$, where C is an arbitrary constant, and

$$y(x) = C\int_\Gamma \mathrm{e}^{x\zeta+\zeta^3/3}\,\mathrm{d}\zeta. \qquad (5.54)$$

Now the integrand decays at infinity in three sectors: either $\pi/6 < \arg\zeta < \pi/2$, or $5\pi/6 < \arg\zeta < 7\pi/6$, or $-\pi/2 < \arg\zeta < -\pi/6$. Therefore, the integration contour Γ should start and end as $\zeta \to \infty$ in one of these sectors. Since the integrand in (5.54) is entire, the integral will be nonzero only if Γ begins and ends in two different sectors. Therefore there are three possibilities for Γ, as shown in Figure 5.10, leading to three distinct solutions for $y(x)$. However, by contour deformation we can write

$$\int_{\Gamma_1} \mathrm{e}^{x\zeta+\zeta^3/3}\,\mathrm{d}\zeta + \int_{\Gamma_2} \mathrm{e}^{x\zeta+\zeta^3/3}\,\mathrm{d}\zeta = \int_{\Gamma_3} \mathrm{e}^{x\zeta+\zeta^3/3}\,\mathrm{d}\zeta, \qquad (5.55)$$

so there are only two independent solutions, as expected for the second-order linear ODE (5.51).

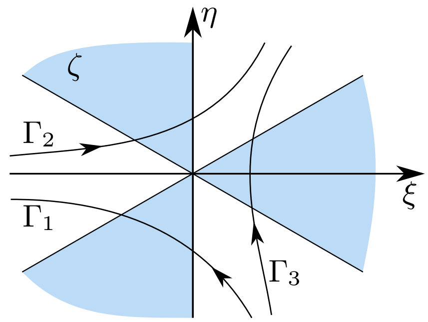

Figure 5.10: Three integration contours for Airy's equation.

This method is equivalent to formally taking a Fourier transform of the equation, and then choosing an inversion contour so that the resulting solution exists. Note that if there was a coefficient of x^2 in the ODE, then integration by parts would lead to a second-order ODE for $g(\zeta)$, which might be just as hard to solve as the original ODE for $y(x)$. The coefficient of x in the Airy equation (5.51) produced a first-order ODE (5.53) for $g(\zeta)$ and thus apparently a single integral solution (5.54). However, the freedom in the choice of the integration contour Γ gave us the required two independent solutions.