6 The Wiener-Hopf method

6.1 Example: an ODE problem

We will first illustrate the ideas underlying the Wiener–Hopf method by using it to solve a simple ODE problem, before moving onto more complicated integral equations and mixed-boundary-value problems. Suppose the complex-valued function $y(x)$ satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b^2 y = 0 \quad \text{for } x < 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a^2 y = 0 \quad \text{for } x > 0, \tag{6.1a}$$

with

$$y(0+) - y(0-) = 0, \qquad \frac{\mathrm{d}y}{\mathrm{d}x}(0+) - \frac{\mathrm{d}y}{\mathrm{d}x}(0-) = 1, \tag{6.1b}$$

and $y(x) \to 0$ as $|x| \to \infty$, where $a, b \in \mathbb{C}$ with $a \neq b$ and $\operatorname{Im} a > 0$ and $\operatorname{Im} b > 0$. The problem (6.1) may readily be solved via elementary methods; here we will show how to obtain the solution by taking a complex Fourier transform and using analytic continuation.

We begin by assuming that $y(x) = O\left(\mathrm{e}^{\alpha x}\right)$ as $x \to \infty$ and $y(x) = O\left(\mathrm{e}^{\beta x}\right)$ as $x \to -\infty$ for some real constants $\alpha < \beta$ whose existence we shall need to verify *a posteriori* (once we have found the solution). We then define the half-range functions

$$y_+(x) = \begin{cases} 0 & x < 0, \\ y(x) & x > 0, \end{cases} \qquad y_-(x) = \begin{cases} y(x) & x < 0, \\ 0 & x > 0, \end{cases} \tag{6.2}$$

so that

$$\bar{y}_+(k) = \int_0^\infty y(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \tag{6.3}$$

is holomorphic in $\operatorname{Im} k > \alpha$, and

$$\bar{y}_-(k) = \int_{-\infty}^0 y(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \tag{6.4}$$

is holomorphic in $\operatorname{Im} k < \beta$.

Integration by parts gives

$$\begin{aligned} 0 &= \int_0^\infty \left(y''(x) + a^2 y(x)\right)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \\ &= -y'(0+) + \mathrm{i}ky(0+) + \left(a^2 - k^2\right)\bar{y}_+(k), \end{aligned} \tag{6.5}$$

provided $\bar{y}_+(k)$ exists and $(y'(x) - \mathrm{i}ky(x))\mathrm{e}^{\mathrm{i}kx} \to 0$ as $x \to \infty$, which is the case for $\operatorname{Im} k > \alpha$. Similarly,

$$0 = y'(0-) - \mathrm{i}ky(0-) + (b^2 - k^2)\bar{y}_-(k) \tag{6.6}$$

provided $\operatorname{Im} k < \beta$. Using the jump conditions (6.1b) at $x = 0$ to eliminate the unknowns $y(0\pm)$ and $y'(0\pm)$ gives

$$\left(k^2 - a^2\right) \bar{y}_+(k) + \left(k^2 - b^2\right) \bar{y}_-(k) = -1, \tag{6.7}$$

for $\alpha < \operatorname{Im} k < \beta$. This is just one equation relating the two unknown functions $\bar{y}_-(k)$ and $\bar{y}_+(k)$. However, remarkably there is enough auxiliary information to determine both functions, using the *Wiener–Hopf method*.

Provided $k \neq a$ and $k \neq -b$, we can divide (6.7) through by $(k-a)(k+b)$ and rearrange to obtain

$$\left(\frac{k+a}{k+b}\right) \bar{y}_+(k) + \left(\frac{k-b}{k-a}\right) \bar{y}_-(k) = -\frac{1}{(k-a)(k+b)} = \frac{1}{a+b}\left(\frac{1}{k+b} - \frac{1}{k-a}\right) \tag{6.8}$$

so that

$$\left(\frac{k+a}{k+b}\right) \bar{y}_+(k) - \frac{1}{(a+b)(k+b)} = -\left(\frac{k-b}{k-a}\right) \bar{y}_-(k) - \frac{1}{(a+b)(k-a)}. \tag{6.9}$$

The left-hand side of (6.9) is holomorphic in $\operatorname{Im} k > \alpha_1 = \max(\alpha, -\operatorname{Im} b)$, while the right-hand side is holomorphic in $\operatorname{Im} k < \beta_1 = \min(\beta, \operatorname{Im} a)$. Since $\alpha < \beta$, by assumption, and $-\operatorname{Im} b < 0 < \operatorname{Im} a$, it follows that $\alpha_1 < \beta_1$ so that these half-planes intersect in the **overlap strip** $\alpha_1 < \operatorname{Im} k < \beta_1$. Hence, the left- and right-hand sides of (6.9) are equal on a dense set of points, and therefore the right-hand side is the analytic continuation of the left-hand side into the lower half-plane, so together they define an entire function, $E(k)$, that is

$$\left(\frac{k+a}{k+b}\right) \bar{y}_+(k) - \frac{1}{(a+b)(k+b)} = -\left(\frac{k-b}{k-a}\right) \bar{y}_-(k) - \frac{1}{(a+b)(k-a)} = E(k). \tag{6.10}$$

Since $\bar{y}_+(k) \to 0$ as $k \to \infty$ in $\operatorname{Im} k > \alpha_1$ and $\bar{y}_-(k) \to 0$ as $k \to \infty$ in $\operatorname{Im} k < \beta_1$, we deduce that $E(k) \to 0$ as $k \to \infty$ and hence by Liouville's theorem, $E(k) \equiv 0$. Thus we obtain unique solutions for *both* functions $\bar{y}_\pm(k)$, namely

$$\bar{y}_+(k) = \frac{1}{(a+b)(k+a)}, \qquad \bar{y}_-(k) = -\frac{1}{(a+b)(k-b)}, \tag{6.11}$$

which may easily be inverted, recalling that the inversion contour must pass *above* the pole $k = -a$ in $y_+(k)$ and *below* the pole $k = b$ in $y_-(k)$.

Thus we obtain the solution

$$y(x) = \begin{cases} \dfrac{\mathrm{e}^{-\mathrm{i}bx}}{\mathrm{i}(a+b)} & x < 0, \\ \dfrac{\mathrm{e}^{\mathrm{i}ax}}{\mathrm{i}(a+b)} & x > 0, \end{cases} \tag{6.12}$$

and we can confirm *a posteriori* our initial assumptions: $y(x) = O\left(\mathrm{e}^{\alpha x}\right)$ as $x \to \infty$ and $y(x) = O\left(\mathrm{e}^{\beta x}\right)$ as $x \to -\infty$, where $\alpha = -\operatorname{Im} a$ and $\beta = \operatorname{Im} b$, so that $\alpha < \beta$, as required.

6.2 The Wiener-Hopf method

The canonical Wiener–Hopf problem is to find functions $\bar{u}_+(k)$ holomorphic in $\operatorname{Im} k > \alpha$ and $\bar{v}_-(k)$ holomorphic in $\operatorname{Im} k < \beta$ such that

$$F(k)\bar{u}_+(k) + \bar{v}_-(k) = G(k) \quad \text{on } \Omega = \{k \in \mathbb{C} : \alpha < \operatorname{Im} k < \beta\}, \tag{6.13}$$

where $F(k)$, $G(k)$ are prescribed holomorphic functions and $F(k) \neq 0$ on Ω. Here, as in the example from §6.1, we use the subscript $_+$ to indicate a function that is holomorphic on some upper half-plane, and $_-$ to indicate a function holomorphic on a lower half-plane.

The general Wiener–Hopf method for solving this problem is as follows.

1. ***Product* decomposition of $F(k)$**
   Find $M_+(k)$ holomorphic in $\operatorname{Im} k > \alpha_1$ and $N_-(k)$ holomorphic in $\operatorname{Im} k < \beta_1$ such that $\alpha \leq \alpha_1 < \beta_1 \leq \beta$ and

   $$F(k) = \frac{M_+(k)}{N_-(k)} \quad \text{on } \Omega_1 = \{k \in \mathbb{C} : \alpha_1 < \operatorname{Im} k < \beta_1\} \subseteq \Omega. \tag{6.14}$$

   We can assume that $M_+(k) \neq 0$ and $N_-(k) \neq 0$ on Ω_1 by cancelling common factors. If we can find such functions $M_+(k)$ and $N_-(k)$, then we can transform (6.13) to

   $$M_+(k)\bar{u}_+(k) + N_-(k)\bar{v}_-(k) = N_-(k)G(k) \quad \text{on } \Omega_1. \tag{6.15}$$

2. ***Sum* decomposition of $N_-(k)G(k)$**
   Find $P_+(k)$ holomorphic in $\operatorname{Im} k > \alpha_2$ and $Q_-(k)$ holomorphic in $\operatorname{Im} k < \beta_2$ such that $\alpha_1 \leq \alpha_2 < \beta_2 \leq \beta_1$ and

   $$N_-(k)G(k) = P_+(k) + Q_-(k) \quad \text{on } \Omega_2 = \{k \in \mathbb{C} : \alpha_2 < \operatorname{Im} k < \beta_2\} \subseteq \Omega_1. \tag{6.16}$$

   Given the functions $P_+(k)$ and $Q_-(k)$, we can then transform (6.15) to

   $$M_+(k)\bar{u}_+(k) + N_-(k)\bar{v}_-(k) = P_+(k) + Q_-(k) \quad \text{on } \Omega_2. \tag{6.17}$$

3. **Analytic continuation**
   Define
   $$E(k) = M_+(k)\bar{u}_+(k) - P_+(k) = Q_-(k) - N_-(k)\bar{v}_-(k) \quad \text{on } \Omega_2. \tag{6.18}$$

   Since the two expressions for $E(k)$ are equal on a dense set of points Ω_2, we can combine them to analytically continue $E(k)$ outside Ω_2 as follows:

   $$E(k) = \begin{cases} M_+(k)\bar{u}_+(k) - P_+(k) & \operatorname{Im} k > \alpha_2, \\ Q_-(k) - N_-(k)\bar{v}_-(k) & \operatorname{Im} k < \beta_2. \end{cases} \tag{6.19}$$

   The resulting function $E(k)$ is therefore entire.

4. **Behaviour at infinity**
   Since $M_+(k)$, $P_+(k)$, $N_-(k)$, and $Q_-(k)$ are in principle known functions of k, we know their behaviour as $|k| \to \infty$.

   The behaviour of $\bar{u}_+(k)$ as $k \to \infty$ in $\operatorname{Im} k > \alpha_2$ and the behaviour of $\bar{v}_-(k)$ as $k \to \infty$ in $\operatorname{Im} k < \beta_2$ are in principle determined by the behaviours of $u_+(x)$ and $v_-(x)$ as $x \to 0_\pm$.

In general this step relies on asymptotic analysis of the relevant Fourier integrals, but in this course we will assume that the required behaviour is given.

Then we know the behaviour of $E(k)$ as $|k| \to \infty$, and we can then apply Liouville's theorem as follows.

- If $E(k) \to$ constant as $|k| \to \infty$, then $E(k)$ is entire and bounded, and therefore a constant by Liouville's theorem. In particular if $E(k) \to 0$ as $|k| \to \infty$, then $E(k) \equiv 0$.
- If $E(k) = O(k^n)$ as $|k| \to \infty$, where $n \in \mathbb{N}$, then $E(k)$ is a polynomial of degree n.

To solve the system for $n \geq 0$, we therefore need to determine $n+1$ coefficients from the boundary conditions. This means that in practice we want n to be as small as possible.

5. **Invert**
To find $u_+(x)$ and $v_-(x)$, we need to apply Fourier inversion to

$$\bar{u}_+(k) = \frac{P_+(k) + E(k)}{M_+(k)} \qquad \text{and} \qquad \bar{v}_-(k) = \frac{Q_-(k) - E(k)}{N_-(k)}. \tag{6.20}$$

In principle, analytic continuation allows the inversion contours to be deformed outside Ω_2 but they must still pass *above* the singularities in $P_+(k)$, $M_+(k)$, and *below* the singularities in $Q_-(k)$, $N_-(k)$.

Remarks

- Steps 1 and 2 in the Wiener–Hopf method are called **Wiener–Hopf decompositions** (product and sum, respectively). These are not unique, for example, the sum decomposition (6.16) would be unaffected if we add any entire function of k to $P_+(k)$ and subtract the same entire function from $Q_-(k)$. Our aim is to find the decomposition that makes analysing the behaviour at infinity (step 4) as straightforward as possible.

- In many applications we can spot the decompositions, though we will describe a constructive method below.

Wiener–Hopf decomposition

General method for sum decomposition

Suppose $G(z)$ is holomorphic in Ω where $\Omega = \{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \beta\}$ and $G(z) \to 0$ as $z \to \infty$ in Ω. By Cauchy's integral formula we can write $G(z)$ as

$$G(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{G(\zeta)}{\zeta - z} \, \mathrm{d}\zeta, \tag{6.21}$$

where $\Gamma \in \Omega$ is the contour illustrated in Figure 6.1, whose interior is the rectangular region $-R < \operatorname{Re} z < R$, $\alpha < \gamma_+ < \operatorname{Im} z < \gamma_- < \beta$. Since $G(\zeta) \to 0$ as $|\zeta| \to \infty$ in Ω, the contribution from the vertical sides tends to zero as $R \to \infty$, giving

$$G(z) = G_+(z) - G_-(z) \quad \text{for } \gamma_+ < \operatorname{Im} z < \gamma_-, \tag{6.22}$$

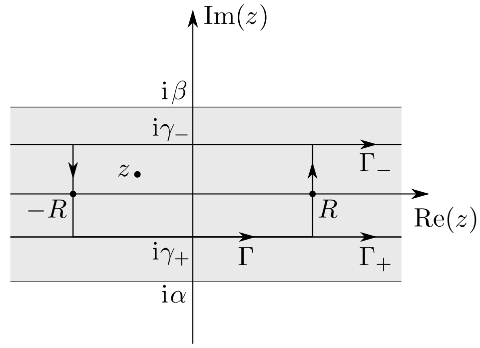


Figure 6.1: Integration contour for sum decomposition.

where

$$G_\pm(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\pm} \frac{G(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \tag{6.23}$$

and $\Gamma_\pm = \{x + \mathrm{i}\gamma_\pm : -\infty < x < \infty\}$. Note that Γ_+ passes underneath z, while Γ_- passes above z. Since $G_\pm(z)$ is holomorphic everywhere except on $\Gamma_\pm$, we deduce that $G_+(z)$ is holomorphic in $\operatorname{Im} z > \gamma_+$ and $G_-(z)$ is holomorphic in $\operatorname{Im} z < \gamma_-$.

General method for product decomposition

Here the trick is to take logs to turn product decomposition into sum decomposition. Suppose $F(z)$ is holomorphic in Ω, that $F(z) \neq 0$ on Ω and that $F(z) \to 1$ as $z \to \infty$ in Ω, where Ω is again the strip $\{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \beta\}$. Since $F(z)$ is non-zero, $\log F(z)$ is holomorphic on Ω and $\log F(z) \to 0$ as $z \to \infty$ in Ω. Hence, we can set $G(z) = \log F(z)$ in (6.22) to obtain

$$\log F(z) = G_+(z) - G_-(z) \quad \text{for } \gamma_+ < \operatorname{Im} z < \gamma_-. \tag{6.24}$$

Defining $F_\pm(z) = \mathrm{e}^{G_\pm(z)}$ on Ω, we deduce that

$$F(z) = \frac{\mathrm{e}^{G_+(z)}}{\mathrm{e}^{G_-(z)}} = \frac{F_+(z)}{F_-(z)} \quad \text{for } \gamma_+ < \operatorname{Im} z < \gamma_-, \tag{6.25}$$

where

$$F_\pm(z) = \exp\left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\pm} \frac{\log F(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \right) \quad \text{for } \gamma_+ < \operatorname{Im} z < \gamma_-, \tag{6.26}$$

and $F_\pm(z)$ are holomorphic on $\operatorname{Im} z > \gamma_+$ and on $\operatorname{Im} z < \gamma_-$ respectively.

Example. Suppose we wish to find a product decomposition of

$$F(z) = \frac{z^2 - a^2}{z^2 - b^2} = \frac{(z-a)(z+a)}{(z-b)(z+b)} \tag{6.27}$$

where $a \neq b \in \mathbb{C}$, with $\operatorname{Im} a > 0$ and $\operatorname{Im} b > 0$. With the overlap region $-\gamma < \operatorname{Im} z < \gamma$, where $\gamma = \min(\operatorname{Im} a, \operatorname{Im} b)$, as shown in Figure 6.2, we find by inspection that

$$F(z) = \frac{F_+(z)}{F_-(z)}, \qquad \text{where} \quad F_+(z) = \frac{z+a}{z+b}, \qquad F_-(z) = \frac{z-b}{z-a}. \tag{6.28}$$

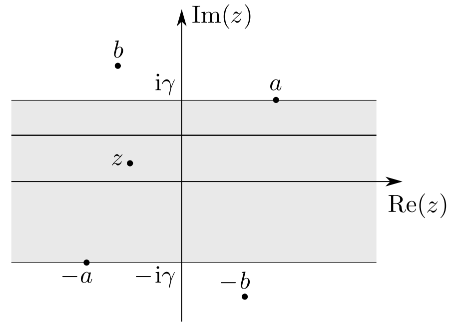


Figure 6.2: The overlap region $-\gamma < \operatorname{Im} z < \gamma$ for $F(z) = \left(z^2 - a^2\right) / \left(z^2 - b^2\right)$.

Thus $F_+(z)$ is holomorphic in $\operatorname{Im} z > -\gamma$ and $F_-(z)$ is holomorphic in $\operatorname{Im} z < \gamma$.

Clearly the factorisation (6.28) is not unique: We could for example multiply both $F_-(z)$ and $F_+(z)$ by any entire function. However, the decomposition (6.28) is the only one for which the overlap region includes $\operatorname{Im} z = 0$ and such that $F_\pm(z) \to 1$ as $z \to \infty$. If we multiplied through by a nontrivial entire function, for example a polynomial in z, the behaviour of $F_\pm(z)$ at infinity would be more complicated.

Now we show how the same decomposition may be found from the general formula (6.26). We write

$$\log F(z) = \left\{\log(z-a) - \log(z-b)\right\} + \left\{\log(z+a) - \log(z+b)\right\}, \tag{6.29}$$

and then consider each of the terms in turn. First consider

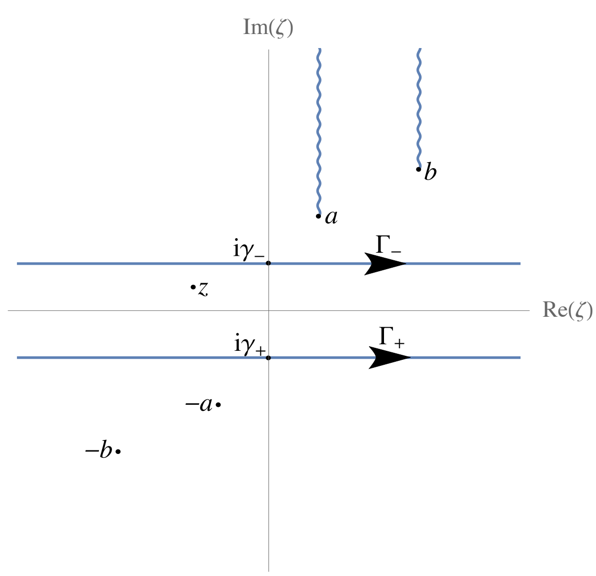


Figure 6.3: Integration contours for the integrals $I_\pm(z)$ defined in equation (6.30).

$$I_\pm(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty+\mathrm{i}\gamma_\pm}^{\infty+\mathrm{i}\gamma_\pm} \frac{\log(\zeta-a) - \log(\zeta-b)}{\zeta - z}\, \mathrm{d}\zeta, \tag{6.30}$$

where $-\gamma < \gamma_+ < \operatorname{Im} z < \gamma_- < \gamma$. As shown in Figure 6.3, the integrand in (6.30) has singularities at $\zeta = a$, $\zeta = b$, but may be analytically continued into $\operatorname{Im} \zeta < \gamma_-$ except for the pole at $\zeta = z$. By closing the contours $\Gamma_\pm$ at $-\mathrm{i}\infty$ and using Cauchy's Residue Theorem, we deduce that

$$I_+(z) = 0, \qquad I_-(z) = -\log(z-a) + \log(z-b), \tag{6.31}$$

where the minus sign comes from the clockwise orientation of the integration contour. Similarly, with

$$J_\pm(z) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty+\mathrm{i}\gamma_\pm}^{\infty+\mathrm{i}\gamma_\pm} \frac{\log(\zeta+a) - \log(\zeta+b)}{\zeta - z}\,\mathrm{d}\zeta, \tag{6.32}$$

the branch points are at $\zeta = -a, -b$, and we can close the integration contours at $+\mathrm{i}\infty$ to get

$$J_+(z) = \log(z+a) - \log(z+b), \qquad J_-(z) = 0, \tag{6.33}$$

Combining (6.30) and (6.32), we get

$$\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_+} \frac{\log F(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \log(z+a) - \log(z+b), \tag{6.34a}$$

$$\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_-} \frac{\log F(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = -\log(z-a) + \log(z-b), \tag{6.34b}$$

and therefore the general formula (6.26) produces

$$F_+(z) = \frac{z+a}{z+b}, \qquad F_-(z) = \frac{z-b}{z-a}, \tag{6.35}$$

in agreement with the previously spotted decomposition (6.28).

6.3 Wiener–Hopf applied to an integral equation

Problem. Find a smooth bounded function $f(x)$ such that

$$\int_0^\infty K(x-t) f(t)\,\mathrm{d}t = f(x) \quad \text{for } x \geq 0, \tag{6.36}$$

where $K(x) = \mathrm{e}^{-|x|}$ for $-\infty < x < \infty$.

Solution. If this were a full range integral equation ($-\infty < x < \infty$), then we could solve it easily by taking a Fourier transform. Our first step is thus to express (6.36) as a full-range integral equation. By defining

$$f_+(x) = \begin{cases} 0 & x < 0, \\ f(x) & x > 0, \end{cases} \qquad h_-(x) = \begin{cases} \displaystyle\int_0^\infty K(x-t) f(t)\,\mathrm{d}t & x < 0, \\ 0 & x > 0, \end{cases} \tag{6.37}$$

we can rewrite (6.36) as

$$\int_{-\infty}^\infty K(x-t) f_+(t)\,\mathrm{d}t = f_+(x) + h_-(x) \quad \text{for } -\infty < x < \infty. \tag{6.38}$$

Now we can take the Fourier transform of the full-range equation (6.38) and use the Convolution Theorem to get

$$\bar{K}(k)\,\bar{f}_+(k) = \bar{f}_+(k) + \bar{h}_-(k). \tag{6.39}$$

Since $f(x)$ is assumed to be bounded, $\bar{f}_+(k)$ is holomorphic in $\operatorname{Im} k > 0$. Since, for $x < 0$,

$$h_-(x) = \mathrm{e}^x \int_0^\infty \mathrm{e}^{-t} f(t)\,\mathrm{d}t = O\left(\mathrm{e}^x\right) \quad \text{as } x \to -\infty, \tag{6.40}$$

it follows that $\bar{h}_-(k)$ is holomorphic in $\operatorname{Im} k < 1$. Hence both $\bar{f}_+(k)$ and $\bar{h}_-(k)$ are holomorphic in the overlap strip $0 < \operatorname{Im} k < 1$, and the problem (6.39) may be solved by the Wiener–Hopf technique.

First we calculate

$$\bar{K}(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-|x|+\mathrm{i}kx}\,\mathrm{d}x = \frac{2}{1+k^2}, \tag{6.41}$$

and substitute into (6.39) to get

$$\frac{1-k^2}{1+k^2}\,\bar{f}_+(k) = \bar{h}_-(k) \quad \text{for } 0 < \operatorname{Im} k < 1. \tag{6.42}$$

We factorise $\left(1-k^2\right)/\left(1+k^2\right)$ using

$$\frac{1-k^2}{1+k^2} = \frac{K_+(k)}{K_-(k)}, \qquad \text{where } K_+(k) = \frac{1-k^2}{k+\mathrm{i}}, \qquad K_-(k) = k - \mathrm{i}, \tag{6.43}$$

so that $K_+(k)$ is holomorphic in $\operatorname{Im} k > -1$ and $K_-(k)$ is entire. Hence,

$$\frac{1-k^2}{k+\mathrm{i}}\bar{f}_+(k) = (k-\mathrm{i})\bar{h}_-(k) = E(k) \text{ (say)} \quad \text{for } 0 < \operatorname{Im} k < 1, \tag{6.44}$$

with the left-hand side holomorphic in $\operatorname{Im} k > 0$ and the right-hand side holomorphic in $\operatorname{Im} k < 1$. Thus the right-hand side of (6.44) is the analytic continuation of the left-hand side of (6.44) into the lower half-plane, so together they define an entire function $E(k)$.

To pin down $E(k)$, we need to consider the behaviour of the functions in equation (6.44) as $k \to \infty$. It may be shown that, since $f(x)$ is assumed to be smooth and bounded on $(0,\infty)$, it follows that $\bar{f}_+(k)$ is $O\left(k^{-1}\right)$ as $k \to \infty$ in $\operatorname{Im} k > 0$. Similarly, assuming that $h_-(x)$ is smooth and bounded on $(-\infty, 0)$, it follows that $h_-(x) = O\left(k^{-1}\right)$ as $k \to \infty$ in $\operatorname{Im} k < 1$. The detailed asymptotic analysis required to prove these results is not required for this course: for completeness a simple derivation is given below.

Given that $\bar{f}_+(k)$ and $\bar{h}_-(k)$ are both $O\left(k^{-1}\right)$ as $k \to \infty$ in their respective half-planes, we deduce from equation (6.44) that $E(k) \to C$, a constant, as $k \to \infty$, and Liouville's theorem implies that that $E(k) \equiv C$. Hence we find

$$\bar{f}_+(k) = \frac{C(k+\mathrm{i})}{1-k^2}, \tag{6.45}$$

so the solution is given by

$$f_+(x) = \frac{C}{2\pi}\int_\Gamma \frac{(k+\mathrm{i})\mathrm{e}^{-\mathrm{i}kx}}{1-k^2}\,\mathrm{d}k, \tag{6.46}$$

where the inversion contour Γ lies in the strip $0 < \operatorname{Im} k < 1$ and thus passes *above* the poles of $\bar{f}_+(k)$ at $k = \pm 1$. By analytic continuation of $\bar{f}_+(k)$ into the lower half-plane, we can close the contour at $-\infty\mathrm{i}$ to obtain the solution

$$f(x) = f_+(x) = -2\pi\mathrm{i}\left(\operatorname{res}\left[\bar{f}_+(k)\mathrm{e}^{-\mathrm{i}kx}; -1\right] + \operatorname{res}\left[\bar{f}_+(k)\mathrm{e}^{-\mathrm{i}kx}; 1\right]\right) = \mathrm{i}C(\cos x + \sin x), \quad (6.47)$$

where C is an arbitrary constant.

Finally, we verify our claim about the asymptotic behaviour of $f_+(k)$ as $k \to \infty$. A simple approach is to assume that $f_+(x)$ is differentiable, with bounded derivative $f_+'(x)$, and then integrate by parts to get

$$\begin{aligned}\bar{f}_+(k) &= \int_0^\infty f_+(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \\ &= \left[\frac{f_+(x)\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}\right]_0^\infty + \frac{\mathrm{i}}{k}\int_0^\infty f_+'(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x. \end{aligned} \quad (6.48)$$

In the first integrated term, the $x = \infty$ limit evaluates to zero provided $\operatorname{Im} k > 0$. In the second term, the integral is bounded, and indeed tends to zero as $k \to \infty$ with $\operatorname{Im} k > 0$ (this is an instance of the *Riemann–Lebesgue Lemma*). Therefore we have

$$\bar{f}_+(k) \sim \frac{\mathrm{i}f_+(0)}{k} + o\left(k^{-1}\right) \quad \text{as } k \to \infty \text{ with } \operatorname{Im} k > 0 \quad (6.49)$$

as required.

6.4 A mixed boundary value problem

The temperature $u(x, y)$ in an inviscid fluid flowing uniformly past a heated semi-infinite plate is governed by the partial differential equation

$$\nabla^2 u = \frac{\partial u}{\partial x} \quad \text{in } y > 0, \quad (6.50\text{a})$$

with the boundary and far field conditions

$$u = 1 \quad \text{on } y = 0,\ x > 0, \qquad \frac{\partial u}{\partial y} = 0 \quad \text{on } y = 0,\ x < 0, \quad (6.50\text{b})$$

$$u \to 0 \quad \text{as } |\boldsymbol{x}| \to \infty, \qquad u = 1 + O\left(|\boldsymbol{x}|^{1/2}\right) \quad \text{as } |\boldsymbol{x}| \to 0, \quad (6.50\text{c})$$

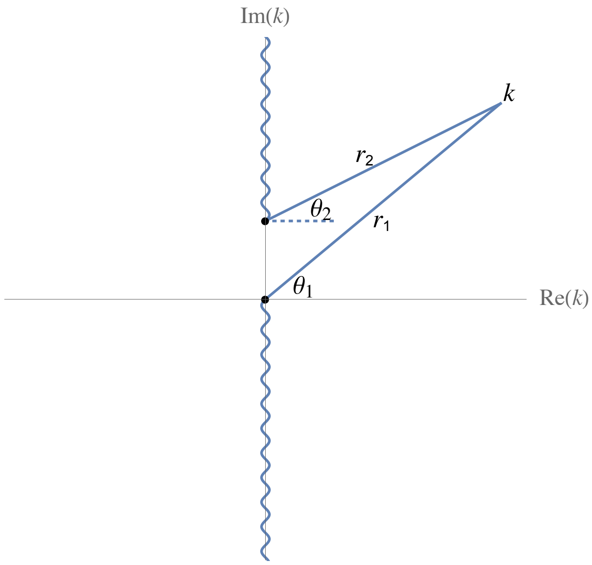
where $\boldsymbol{x} = (x, y)$. Note that this is a *mixed boundary value problem* because the boundary conditions switch from Neumann (specified $\partial u/\partial y$) to Dirichlet (specified u) across $x = 0$. We know from Chapter 4 to expect square root type behaviour at the origin where the boundary condition switches.

Take the Fourier transform of the partial differential equation (6.50a) to give

$$\frac{\partial^2 \bar{u}}{\partial y^2} - (k^2 - \mathrm{i}k)\bar{u} = 0 \quad \text{in } y > 0. \quad (6.51)$$

Since we require $\bar{u}(k, y) \to 0$ as $y \to \infty$, the relevant solution is

$$\bar{u}(k, y) = A(k)\mathrm{e}^{-(k^2 - \mathrm{i}k)^{1/2} y}, \quad (6.52)$$



$$r_1 = |k| \in [0, \infty),$$

$$r_2 = |k - \mathrm{i}| \in [0, \infty),$$

$$\theta_1 = \arg k \in [-\pi/2, 3\pi/2),$$

$$\theta_2 = \arg(k - \mathrm{i}) \in [-3\pi/2, \pi/2).$$

Figure 6.4: The lengths r_1, r_2 and angles θ_1, θ_2 used to define the multifunction $\left(k^2 - \mathrm{i}k\right)^{1/2}$.

where $A(k)$ is an arbitrary integration function, and the branch of the square root is defined such that $\mathrm{Re}\left(k^2 - \mathrm{i}k\right)^{1/2} > 0$ on the inversion contour. Thus we place the branch cut along the imaginary k-axis from $-\infty\mathrm{i}$ to 0 and from i to $\infty\mathrm{i}$, as illustrated in Figure 6.4, and define

$$(k^2 - \mathrm{i}k)^{1/2} = \sqrt{r_1 r_2}\, \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}, \tag{6.53}$$

where the lengths $r_{1,2}$ and angles $\theta_{1,2}$ are as shown in Figure 6.4, so that $\mathrm{Re}(k^2 - \mathrm{i}k)^{1/2} > 0$ everywhere on the cut k-plane.

Now to take care of the mixed boundary conditions on $y = 0$, let

$$g_+(x) = \begin{cases} 0 & x < 0, \\ \dfrac{\partial u}{\partial y}(x, 0) & x > 0, \end{cases} \qquad f_-(x) = \begin{cases} u(x, 0) & x < 0, \\ 0 & x > 0, \end{cases} \tag{6.54}$$

so that (6.50b) may be restated as the full-range boundary conditions

$$u(x, 0) = f_-(x) + \mathrm{H}(x), \qquad \frac{\partial u}{\partial y}(x, 0) = g_+(x), \tag{6.55}$$

where $\mathrm{H}(x)$ is the Heaviside function.

We suppose that $g_+(x) = O\left(\mathrm{e}^{\alpha x}\right)$ as $x \to \infty$ and that $f_-(x) = O\left(\mathrm{e}^{\beta x}\right)$ as $x \to -\infty$ for some constants α, β such that $\alpha < \beta$. Then $\bar{g}_+(k)$ is holomorphic in $\mathrm{Im}\, k > \alpha$ and $\bar{f}_-(k)$ is holomorphic in $\mathrm{Im}\, k < \beta$, so that both functions are holomorphic in the overlap strip $\alpha < \mathrm{Im}\, k < \beta$. We also recall that $\bar{\mathrm{H}}(k) = \mathrm{i}/k$ for $\mathrm{Im}\, k > 0$, so that, provided $\beta > 0$, we can take the Fourier transform of the boundary conditions (6.55) to get

$$\bar{u}(k, 0) = \bar{f}_-(k) + \frac{\mathrm{i}}{k} \quad \text{for } 0 < \mathrm{Im}\, k < \beta, \qquad \frac{\partial \bar{u}}{\partial y}(k, 0) = \bar{g}_+(k) \quad \text{for } \mathrm{Im}\, k > \alpha. \tag{6.56}$$

Now we just substitute in the solution (6.52) for $\bar{u}$:

$$A(k) = \bar{f}_-(k) + \frac{\mathrm{i}}{k}, \qquad -A(k)(k^2 - \mathrm{i}k)^{1/2} = \bar{g}_+(k). \tag{6.57}$$

Elimination of $A(k)$ gives

$$\frac{1}{(k^2 - \mathrm{i}k)^{1/2}}\,\bar{g}_+(k) + \bar{f}_-(k) = -\frac{\mathrm{i}}{k}, \tag{6.58}$$

where $\bar{f}_-(k)$ is holomorphic in $\operatorname{Im} k < \beta$ and $\bar{g}_+(k)$ is holomorphic in $\operatorname{Im} k > \alpha$. Provided $0 \le \alpha < \beta \le 1$, we can apply the Wiener–Hopf method, as follows.

First we split $\left(k^2 - \mathrm{i}k\right)^{1/2}$ to separate the singularities at $k = 0$ and $k = 1$ by setting

$$\left(k^2 - \mathrm{i}k\right)^{1/2} = k^{1/2}\,(k - \mathrm{i})^{1/2}, \qquad k^{1/2} = \sqrt{r_1}\,\mathrm{e}^{\mathrm{i}\theta_1/2} \qquad (k - \mathrm{i})^{1/2} = \sqrt{r_2}\,\mathrm{e}^{\mathrm{i}\theta_2/2}, \tag{6.59}$$

where $r_{1,2}$ and $\theta_{1,2}$ are again as in Figure 6.4. Then (6.58) may be rearranged to

$$\frac{\bar{g}_+(k)}{k^{1/2}} + (k - \mathrm{i})^{1/2}\bar{f}_-(k) = -\mathrm{i}\frac{(k - \mathrm{i})^{1/2}}{k}, \tag{6.60}$$

where the first and second terms are holomorphic in $\operatorname{Im} k > \alpha$ and in $\operatorname{Im} k < \beta$ respectively.

The right-hand side of equation (6.60) has a pole at $k = 0$ and a branch point at $k = \mathrm{i}$. To split these, the trick is to separate out the principal part of the pole as follows:

$$\frac{(k - \mathrm{i})^{1/2}}{k} = \frac{(-\mathrm{i})^{1/2}}{k} + \frac{(k - \mathrm{i})^{1/2} - (-\mathrm{i})^{1/2}}{k}. \tag{6.61}$$

The second term on the right-hand side of (6.61) now has a removable singularity at $k = 0$ and therefore defines a holomorphic function in $\operatorname{Im} k < 1$. Here $(-\mathrm{i})^{1/2}$ is equal to $(k - \mathrm{i})^{1/2}$ evaluated at $k = 0$, namely $\mathrm{e}^{-\mathrm{i}\pi/4}$.

Hence, equation (6.60) may be rearranged to

$$\frac{\bar{g}_+(k)}{k^{1/2}} + \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{k} = -(k - \mathrm{i})^{1/2}\bar{f}_-(k) - \frac{\mathrm{i}(k - \mathrm{i})^{1/2} - \mathrm{e}^{\mathrm{i}\pi/4}}{k} \quad \text{for } \alpha < \operatorname{Im} k < \beta, \tag{6.62}$$

with the left-hand side holomorphic in $\operatorname{Im} k > \alpha$ and the right-hand side holomorphic in $\operatorname{Im} k < \beta$. The right-hand side of (6.62) is the analytic continuation of the left-hand side of (6.62) into the lower half-plane, so together they define an entire function, $E(k)$ say.

To determine $E(k)$, we must consider the behaviours of the left- and right-hand sides of equation (6.62) as $k \to \infty$. Since $u(x, 0)$ is required to be bounded as $x \to 0$, it follows (as in the previous example) that $\bar{f}_-(k) = O\left(k^{-1}\right)$ as $k \to \infty$ with $\operatorname{Im} k < \beta$. However, since $u(x, y)$ has a square root behaviour as $(x, y) \to (0, 0)$, we expect $\partial u/\partial y$ to have an inverse square root singularity, that is

$$g_+(x) = O\left(x^{-1/2}\right) \quad \text{as } x \searrow 0. \tag{6.63}$$

It may be shown (for example using Laplace's method) that the corresponding asymptotic behaviour of the Fourier transform is

$$\bar{g}_+(k) = O\left(k^{-1/2}\right) \quad \text{as } k \to \infty \text{ with } \operatorname{Im} k > \alpha. \tag{6.64}$$

Hence, equation (6.62) implies that $E(k) \to 0$ as $k \to \infty$, and we deduce from Liouville's theorem that $E(k) \equiv 0$.

We then solve for $g_+(k)$ and $f_-(k)$:

$$g_+(k) = -\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{k^{1/2}}, \qquad \bar{f}_-(k) = -\frac{\mathrm{i}}{k} + \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{k(k-\mathrm{i})^{1/2}}, \tag{6.65}$$

and can now verify our assumptions *a posteriori*. Clearly $g_+(k)$ is holomorphic in $\operatorname{Im} k > 0$ and is order $k^{-1/2}$ as $k \to \infty$. Since the singularity at $k = 0$ is removable, $f_-(k)$ is holomorphic in $\operatorname{Im} k < 1$ and is order k^{-1} as $k \to \infty$. Thus the required overlap region does exist; the minimum admissible value of α is 0 and the maximum admissible value of β is 1.

To complete the solution, we substitute back into (6.57) to get

$$A(k) = \bar{f}_-(k) + \frac{\mathrm{i}}{k} = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{k(k-\mathrm{i})^{1/2}}, \tag{6.66}$$

and therefore

$$\bar{u}(k, y) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{k(k-\mathrm{i})^{1/2}}\, \mathrm{e}^{\left(k^2-\mathrm{i}k\right)^{1/2} y}. \tag{6.67}$$

Hence the inversion formula gives

$$u(x, y) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{2\pi} \int_\Gamma \frac{\mathrm{e}^{-\left(k^2-\mathrm{i}k\right)^{1/2} y - \mathrm{i}kx}}{k(k-\mathrm{i})^{1/2}}\, \mathrm{d}k, \tag{6.68}$$

where the contour Γ must pass between the singularities at $k = 0$ and at $k = \mathrm{i}$.

Remarkably, the integral (6.68) may be evaluated to obtain

$$u(x, y) = \operatorname{erfc}\left(\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right), \qquad \text{where } \operatorname{erfc}(t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^t \mathrm{e}^{-\xi^2}\, \mathrm{d}\xi \tag{6.69}$$

is the complementary error function, One can readily verify that $u(x, y)$ satisfies the problem (6.50) and has the expected square root behaviour as $(x, y) \to (0, 0)$.

The solution (6.69) may be expressed in the form

$$u(x, y) = \operatorname{erfc}\left(\operatorname{Im} z^{1/2}\right), \quad \text{where } z = x + \mathrm{i}y, \tag{6.70}$$

and in fact this solution may be obtained much more directly by first conformally mapping the problem using the map $z \mapsto \zeta = z^{1/2}$.