Math C5.4, Networks, University of Oxford
Problem Sheet 3

Renaud Lambiotte, Mathematical Institute

1. *Graph Laplacians.* Consider an unweighted, undirected, simple network. Show that the smallest eigenvalue of the combinatorial graph Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is 0. How can one use the spectrum of the graph Laplacian to determine the number of components in the network? Do you have any ideas about how one might think about a graph that is "almost" separated into two disjoint components (and how one might measure how close the components are to being disconnected)?

2. *Generalising clustering coefficients.* The clustering coefficient is classically defined for unweighted, undirected networks without self-edges or multi-edges. Things become more complicated in more general situations, and it is instructive to think about them. How would you define a clustering coefficient for a weighted, undirected network? How would you define a clustering coefficient for an unweighted, undirected bipartite network?

3. *Modularity.*

   (a) Apply modularity optimization techniques implemented in the library of your choice on some examples and visualise the results.

   (b) Ex.VI.2 : Write a function that takes a graph and its partition as an input and returns its modularity. Verify the values obtained in the previous exercise.

   (c) Ex.VI.4 : In the Louvain method, the efficiency of the algorithm partly resides in the fact that the variation of modularity Δ_{ij} obtained by moving a vertex i from its community to the community of one of its neighbors j can be calculated with only local information. In practice, the variation of modularity is calculated by removing i from its community $\Delta_{remove;i}$ (this is only done once) then inserting it into the community of j $\Delta_{insert;ij}$ for each neighbor j of i. The variation is therefore: $\Delta_{ij} = \Delta_{remove;i} + \Delta_{insert;ij}$. Derive analytically $\Delta_{remove;i}$ when removing node i from its community C_i.

   (d) Is it possible that the Louvain method produces communities that do not form connected components?

4. *Critically reading journal articles.* Read Santo Fortunato, Marc Barthelemy, Proc. Natl. Acad. Sci. USA 104 (1), 36-41 (2007) and write a 1-page summary, typeset using LaTeX, of the main findings and methods of this article, as well as its strength and limitations.