4 Plane strain

4.1 Definition

A more common configuration than that of antiplane strain is *plane strain*, in which a solid is displaced in the (x, y)-plane only, with the displacement being independent of z. Writing

$$\boldsymbol{u} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ 0 \end{pmatrix}, \tag{4.1}$$

we find that the stress tensor takes the form

$$\mathcal{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix}, \tag{4.2}$$

where

$$\tau_{xx} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}, \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{yy} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}, \tag{4.3}$$

and

$$\tau_{zz} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{4.4}$$

This configuration arises, for example, when a z-independent traction just in the (x, y)-plane is applied to the curved boundary of a cylindrical bar aligned with the z-axis. The easily-forgotten stress component τ_{zz} represents the normal traction that would need to be applied to the ends of such a bar to prevent it expanding or contracting in the z-direction.

4.2 Compatibility

In steady plane strain, the momentum equation takes the form

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = -\rho g_x, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = -\rho g_y, \tag{4.5}$$

where the gravitational acceleration is $\boldsymbol{g} = (g_x, g_y, 0)^{\mathrm{T}}$. Viewed as a system of equations for the stress components, (4.5) is *under-determined*, comprising just two equations for three unknowns. It is only by using the constitutive relations (4.3) that we are able to obtain a closed system of two equations for the two components (u, v) of displacement.

It follows that, given τ_{ij}, (4.3) is itself an *over-determined* system for the displacements. If we were in the very fortunate position of knowing the stress components τ_{xx}, τ_{xy} and τ_{yy}, which of course must satisfy (4.5), then we could view (4.3) as a system of three equations for the two displacement components (u, v). Whenever u is a twice continuously differentiable single-valued function, we must have $\partial^2 u/\partial x \partial y \equiv \partial^2 u/\partial y \partial x$ and similarly for v. It then follows from cross-differentiation that

$$\frac{\partial^2 \tau_{yy}}{\partial x^2} - 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{xx}}{\partial y^2} = \nu \nabla^2 \left(\tau_{xx} + \tau_{yy}\right). \tag{4.6}$$

This is the *compatibility condition* which ensures that (4.3) can be solved for single-valued functions u and v.

If τ_{ij} does not satisfy (4.6), then there is no two-dimensional displacement field that gives rise to such a stress and it is, therefore, *incompatible* with plane strain. Hence, if we are presented with an elastic material in a state of plane strain under the action of, say, some nonzero boundary tractions, then the necessary and sufficient condition for the material to return to its pristine unstrained state when the tractions are removed is that the stress field in it satisfies (4.6).

4.3 The Airy stress function

When there is no body force, we can guarantee that the stress components satisfy (4.5) by introducing an *Airy stress function* $\mathfrak{A}$ such that

$$\tau_{xx} = \frac{\partial^2 \mathfrak{A}}{\partial y^2}, \qquad \tau_{xy} = -\frac{\partial^2 \mathfrak{A}}{\partial x \partial y}, \qquad \tau_{yy} = \frac{\partial^2 \mathfrak{A}}{\partial x^2}. \tag{4.7}$$

In the same way that the stress function ϕ in antiplane strain is only defined to within a constant, $\mathfrak{A}$ is only defined to within a linear function of x and y; in other words, any such function may be added to $\mathfrak{A}$ without contributing to the stress.

By substituting (4.7) into the compatibility condition (4.6), we find that $\mathfrak{A}$ satisfies the *biharmonic equation*

$$\nabla^4 \mathfrak{A} = 0, \tag{4.8}$$

where

$$\nabla^4 \mathfrak{A} = \nabla^2 \left(\nabla^2 \mathfrak{A}\right) = \frac{\partial^4 \mathfrak{A}}{\partial x^4} + 2\frac{\partial^4 \mathfrak{A}}{\partial x^2 \partial y^2} + \frac{\partial^4 \mathfrak{A}}{\partial y^4} \tag{4.9}$$

is the two-dimensional *biharmonic operator*.

Recall that the general solution of Laplace's equation in two dimensions may be written in the form

$$\phi = \mathrm{Re}\left\{f(z)\right\}, \tag{4.10}$$

where f is an arbitrary analytic function of $z = x + \mathrm{i}y$. Similarly, the general solution of (4.8) has the *Goursat representation*

$$\mathfrak{A} = \mathrm{Re}\left\{\bar{z} f(z) + g(z)\right\}, \tag{4.11}$$

where $\bar{z} = x - \mathrm{i}y$, and f, g are analytic. Many of the other solution techniques that work well on Laplace's equation, for example separation of variables, can be adapted to the biharmonic equation. However, fitting the boundary conditions may be significantly more difficult; we will discuss the boundary conditions to be applied to (4.8) below.

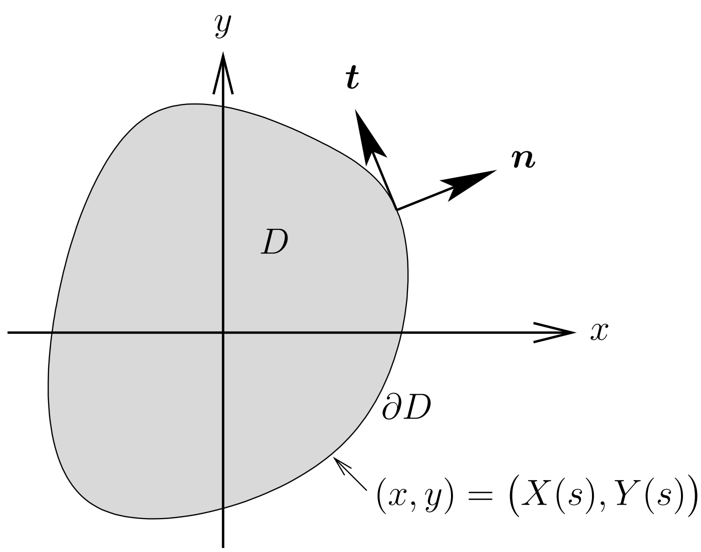

Figure 4.1: Definition sketch showing the unit normal $\boldsymbol{n}$ and tangent $\boldsymbol{t}$ to the boundary ∂D of a plane region D.

Once we have calculated $\mathfrak{A}$, we obtain the following expressions for the displacement gradients from (4.3):

$$2\mu \frac{\partial u}{\partial x} = -\nu \frac{\partial^2 \mathfrak{A}}{\partial x^2} + (1-\nu)\frac{\partial^2 \mathfrak{A}}{\partial y^2}, \tag{4.12a}$$

$$2\mu \frac{\partial v}{\partial y} = (1-\nu)\frac{\partial^2 \mathfrak{A}}{\partial x^2} - \nu \frac{\partial^2 \mathfrak{A}}{\partial y^2}, \tag{4.12b}$$

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2 \mathfrak{A}}{\partial x \partial y}. \tag{4.12c}$$

Given $\mathfrak{A}$ satisfying (4.8), (4.12) forms a compatible system of three equations that determine u and v to within a rigid body displacement in which $u = u_0 - \omega y$, $v = v_0 + \omega x$, with u_0, v_0 and ω constant.

4.4 Boundary conditions

Suppose we wish to solve (4.8) in some region D, on whose boundary a prescribed traction $\boldsymbol{\sigma}$ is imposed, that is

$$\mathcal{T}\boldsymbol{n} = \boldsymbol{\sigma} \qquad \text{on} \quad \partial D. \tag{4.13}$$

As illustrated in Figure 4.1, we parametrise ∂D using $(x, y) = \big(X(s), Y(s)\big)$, where s is arc-length, so the unit tangent and outward normal vectors are given by

$$\boldsymbol{t} = \begin{pmatrix} X' \\ Y' \end{pmatrix}, \qquad \boldsymbol{n} = \begin{pmatrix} Y' \\ -X' \end{pmatrix}, \tag{4.14}$$

where $'$ is shorthand for $\mathrm{d}/\mathrm{d}s$. Using (4.7) to write the stress components in terms of $\mathfrak{A}$, we thus find that (4.13) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \partial\mathfrak{A}/\partial y \\ -\partial\mathfrak{A}/\partial x \end{pmatrix} = \boldsymbol{\sigma}. \tag{4.15}$$

If no surface traction is applied, that is $\boldsymbol{\sigma} = \mathbf{0}$, then it follows from (4.15) that $\boldsymbol{\nabla}\mathfrak{A}$ is constant on ∂D. Since, as noted above, an arbitrary linear function of x and y may be added to $\mathfrak{A}$ without affecting the stresses, we can, in a simply-connected region, take this constant to be zero without loss of generality. Then, by taking the dot product of $\boldsymbol{\nabla}\mathfrak{A}$ with $\boldsymbol{t}$ and $\boldsymbol{n}$ respectively, we deduce that

$$\frac{\mathrm{d}\mathfrak{A}}{\mathrm{d}s} = \frac{\partial\mathfrak{A}}{\partial n} = 0 \tag{4.16}$$

on ∂D. The former of these tells us that $\mathfrak{A}$ is constant on ∂D and, again, this constant may, without loss of generality, be set to zero. Finally, we arrive at the boundary conditions

$$\mathfrak{A} = 0, \qquad \frac{\partial\mathfrak{A}}{\partial n} = 0 \tag{4.17}$$

to be imposed on a stress-free boundary.

We note that the divergence theorem on any closed region D yields

$$\begin{aligned} \int_{\partial D} \left(\mathfrak{A}\frac{\partial}{\partial n}\left(\nabla^2\mathfrak{A}\right) - \frac{\partial\mathfrak{A}}{\partial n}\nabla^2\mathfrak{A} \right) \mathrm{d}s \\ = \iint_D \left\{ \operatorname{div}\left(\mathfrak{A}\operatorname{grad}(\nabla^2\mathfrak{A})\right) - \operatorname{div}\left(\nabla^2\mathfrak{A}\operatorname{grad}\mathfrak{A}\right) \right\} \mathrm{d}x\mathrm{d}y \\ = \iint_D \left\{ \mathfrak{A}\nabla^4\mathfrak{A} - \left(\nabla^2\mathfrak{A}\right)^2 \right\} \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4.18}$$

Hence, if $\mathfrak{A}$ satisfies the biharmonic equation in D and the boundary conditions (4.17) on ∂D, then $\nabla^2\mathfrak{A} = 0$ and a second use of (4.17) reveals that $\mathfrak{A} \equiv 0$. This result confirms that the stresses inside a closed body in plane strain are uniquely determined by the tractions applied to the boundary.

4.5 Plane strain in a disc

As a first illustrative example, let us consider plane strain in a circular region $r < a$ on whose boundary $r = a$ a prescribed traction is applied. The equivalent of (4.7) in plane polar coordinates (r, θ) is

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\mathfrak{A}}{\partial\theta^2} + \frac{1}{r}\frac{\partial\mathfrak{A}}{\partial r}. \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\mathfrak{A}}{\partial\theta}\right), \qquad \tau_{\theta\theta} = \frac{\partial^2\mathfrak{A}}{\partial r^2}, \tag{4.19}$$

and the biharmonic equation for $\mathfrak{A}$ reads

$$\nabla^4\mathfrak{A} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\mathfrak{A} = 0. \tag{4.20}$$

We will only consider here cases where a purely normal pressure P is applied, so the boundary conditions on $r = a$ are

$$\frac{1}{r^2}\frac{\partial^2 \mathfrak{A}}{\partial \theta^2} + \frac{1}{r}\frac{\partial \mathfrak{A}}{\partial r} = -P, \qquad \frac{\partial}{\partial r}\left(\frac{\mathfrak{A}}{r}\right) = 0 \qquad \text{on} \quad r = a; \tag{4.21}$$

we obtain the latter equation by integrating the condition $\tau_{r\theta} = 0$ with respect to θ.

The simplest case occurs if P is constant, so we expect the displacement to be purely radial and $\mathfrak{A}$ to be a function of r alone. The problem thus reduces to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 \mathfrak{A} = 0, \tag{4.22}$$

subject to the boundary conditions

$$\mathfrak{A} = r\frac{\mathrm{d}\mathfrak{A}}{\mathrm{d}r} = -Pa^2 \qquad \text{on} \quad r = a. \tag{4.23}$$

It is straightforward to solve (4.22) in the form

$$\mathfrak{A} = c_1 r^2 + c_2 + c^3 r^2 \log r + c_4 \log r. \tag{4.24}$$

For the stresses to exist throughout the circle, we require $\mathfrak{A}$ to be twice differentiable as $r \to 0$ and hence $c_3 = c_4 = 0$. Then, by using the boundary conditions (4.23), we obtain

$$\mathfrak{A} = -\frac{P}{2}\left(r^2 + a^2\right). \tag{4.25}$$

If P is not assumed to be constant, then we can solve the problem by separating the variables in polar coordinates, using the fact that $\mathfrak{A}$ must be a 2π-periodic function of θ. Seeking a solution of (4.20) in the form

$$\mathfrak{A}(r,\theta) = f(r)\sin(n\theta), \tag{4.26}$$

where n is a positive integer, we find that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n^2}{r^2}\right)^2 f = 0. \tag{4.27}$$

This Euler differential equation admits the solution $f(r) = r^k$, where k satisfies

$$\left(k^2 - n^2\right)\left[(k-2)^2 - n^2\right] = 0. \tag{4.28}$$

We must again ensure that the stress components (4.19) are well defined as $r \to 0$, and now this restricts us to the solutions $k = n, n+2$, that is

$$f(r) = c_1 r^{n+2} + c_2 r^n, \tag{4.29}$$

where the c_i are again arbitrary constants. For the special case $n = 1$, the only physically acceptable solution is

$$f(r) = c_1 r^3. \tag{4.30}$$

(The $c_2 r$ must vanish because of the condition that $\tau_{rr} = [f'(r)/r - n^2 f(r)/r] \sin n\theta$ be well-defined as $r \to 0$.)

We can take a linear combination of separable solutions that satisfy (4.23) and the radially-symmetric solution (4.25) to obtain

$$\mathfrak{A} = -\left(\frac{r^2 + a^2}{4}\right) A_0 + \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{r^n}{(n-1)a^{n-2}} - \frac{r^{n+2}}{(n+1)a^n}\right)\big\{A_n \cos(n\theta) + B_n \sin(n\theta)\big\}, \quad (4.31)$$

where A_n and B_n are the Fourier coefficients of P, that is

$$A_n = \frac{1}{\pi}\int_0^{2\pi} P(\theta)\cos(n\theta)\,\mathrm{d}\theta, \qquad B_n = \frac{1}{\pi}\int_0^{2\pi} P(\theta)\sin(n\theta)\,\mathrm{d}\theta. \quad (4.32)$$

Notice that the $n = 1$ term does not appear in the series in (4.31) because of the condition that $\partial(\mathfrak{A}/r)/\partial r = 0$ at $r = a$. We therefore have that it is possible to satisfy the boundary condition (4.21) only if

$$\int_0^{2\pi} P(\theta)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\mathrm{d}\theta = \mathbf{0}. \quad (4.33)$$

This condition simply states that the net force on the disc must be zero.