7 Contact

7.1 Contact of elastic strings

Contact refers to the class of problems in which two elastic solid bodies are brought into contact with each other, as illustrated in Figure 7.1. The geometrical configuration near the edge of the contact region is apparently similar to that of fracture, with voids now outside the *contact set*, that is the set of points at which the two solids are in contact. (The points where they are not in contact form the *non-contact set*.) The mathematical setup of such problems does therefore have some similarity with fracture, but, in the steady state, there is one crucial difference. Whereas much of the study of fracture concerns cracks of prescribed length, in contact problems the contact set itself is often unknown in advance. They are thus known as *free-boundary problems*, that is problems whose geometry must be determined along with the solution. An example of such a problem is to determine how the contact area between a car tyre and the road depends on the inflation pressure.

We start by considering the simplest elastic contact problem, namely an elastic string making steady contact under a prescribed body force $p(x)$ against a smooth, nearly flat, rigid *obstacle* Γ. If the transverse displacement $w(x)$ is assumed to be small, then a force balance on a small element of the membrane in the non-contact set yields the familiar equation

$$T\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = p(x), \tag{7.1}$$

where the membrane tension T is spatially uniform. Provided friction is negligible, T is also constant throughout the contact set, where w is simply equal to the prescribed obstacle height $f(x)$, which is zero in Figure 7.2.

At the free boundary between the contact and non-contact sets, a local force balance reveals that T and $T\mathrm{d}w/\mathrm{d}x$ must both be continuous. Thus, our task is to solve (7.1) in the

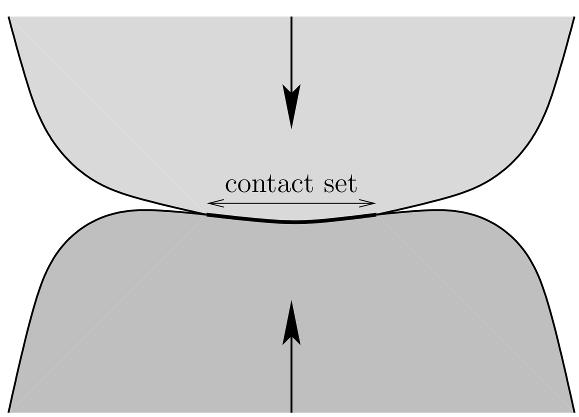

Figure 7.1: Definition sketch for contact between two solids.

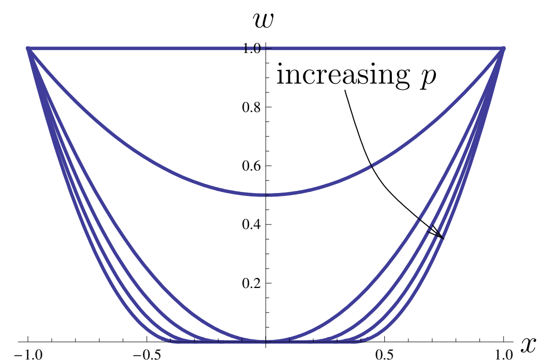

Figure 7.2: Solution for the contact between a string and a level surface, with applied pressure $p/T = 0, 1, 2, 3, 4, 5$.

non-contact set, subject to specified boundary conditions at the ends of the string and

$$w = f, \qquad \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \tag{7.2}$$

at the points where the string meets the obstacle.

Let us illustrate the procedure for a string whose ends $x = \pm 1$ are fixed a unit distance above a flat surface $z = f(x) = 0$. If the applied pressure p is spatially uniform, then we can solve (7.1), subject to the boundary conditions $w(-1) = w(1) = 1$, to obtain

$$w = 1 - \frac{p}{2T}\left(1 - x^2\right). \tag{7.3}$$

As the applied pressure p is increased, the downward displacement at the centre of the string increases, until it first makes contact with $z = 0$ when $p = 2T$.

If the pressure is increased further, then a contact set forms near the middle of the string. Let us denote the boundaries of this region by $x = \pm s$, where s is to be determined as part of the solution. In this simple problem, we only need to solve for positive x; the solution in $x < 0$ may then be inferred by symmetry. We therefore solve (7.1) in $x > s$, subject to $w(1) = 1$ and the continuity conditions

$$w(s) = \frac{\mathrm{d}w}{\mathrm{d}x}(s) = 0 \tag{7.4}$$

to obtain

$$w = 1 - \frac{p}{2T}(1 - x)(1 + x - 2s) \tag{7.5}$$

in $x > s$, where

$$s = 1 - \sqrt{\frac{2T}{p}}. \tag{7.6}$$

In Figure 7.2 we see how, as increasing pressure is applied, the string sags downward, makes contact with the surface beneath, and the contact set then gradually spreads outwards.

This simple example shows how the solution of the contact problem hinges on locating the free boundary between the contact and non-contact sets. Such free boundary problems

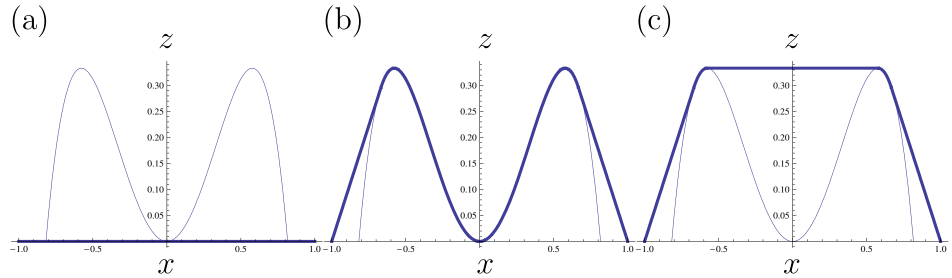

Figure 7.3: Three candidate solutions for the contact problem between a stress-free string and the surface $z = 2x^2 - 3x^4$.

are inevitably nonlinear even when, as in (7.1), the governing equation is itself linear. This means that the solutions of contact problems can exhibit phenomena such as nonuniqueness that are impossible for linear problems.

To illustrate some of the possible pitfalls, consider now a string, subject to zero body force, stretched over the curved surface

$$z = f(x) = 2x^2 - 3x^4 \tag{7.7}$$

with its ends fixed at $w(-1) = w(1) = 0$. With $p = 0$, (7.1) implies that the string must be straight wherever it is out of contact. The most obvious linear function satisfying both boundary conditions is simply $w \equiv 0$ but, as can be seen in Figure 7.3(a), this would imply that the string penetrates the obstacle, which is impossible. We must therefore allow the string to make contact with the obstacle, as illustrated in Figure 7.3(b). Recalling that the string must meet the obstacle tangentially, the solution in this case is easily constructed as

$$w = \begin{cases} \frac{8}{9}(1+x) & -1 \leqslant x \leqslant -\frac{2}{3}, \\ 2x^2 - 3x^4 & -\frac{2}{3} \leqslant x \leqslant \frac{2}{3}, \\ \frac{8}{9}(1-x) & \frac{2}{3} \leqslant x \leqslant 1. \end{cases} \tag{7.8}$$

We can, however, construct a third solution, shown in Figure 7.3(c), in which the string loses contact with the middle of the obstacle. Again noting that the string is linear when out of contact and must always make contact tangentially, we find that this solution takes the form

$$w = \begin{cases} \frac{8}{9}(1+x) & -1 \leqslant x \leqslant -\frac{2}{3}, \\ 2x^2 - 3x^4 & -\frac{2}{3} \leqslant x \leqslant -\frac{1}{\sqrt{3}}, \\ \frac{1}{3} & -\frac{1}{\sqrt{3}} \leqslant x \leqslant \frac{1}{\sqrt{3}}, \\ 2x^2 - 3x^4 & \frac{1}{\sqrt{3}} \leqslant x \leqslant \frac{2}{3}, \\ \frac{8}{9}(1-x) & \frac{2}{3} \leqslant x \leqslant 1. \end{cases} \tag{7.9}$$

We would like to have a mathematical formulation that chooses between possible candidate solutions like those shown in Figure 7.3. The situation shown in Figure 7.3(a) can be avoided by incorporating the requirement $w \geqslant f$ into our model. Although it is less obvious, Figure 7.3(b) is also physically unrealistic, since there is a *tensile* normal stress between the string and the obstacle. In general, if $R(x)$ denotes the normal reaction force exerted on the string by the obstacle, then (7.1) is modified to

$$T\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = p - R. \tag{7.10}$$

The reaction force R must be zero outside the contact set. The solution shown in Figure 7.3(b) has negative values of R and is therefore unphysical; the correct solution is therefore (7.9).

To summarise, we can select a physically relevant solution of the contact problem by insisting that

$$w = f, \quad T\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - p < 0 \qquad \text{in the contact set,} \tag{7.11a}$$

and

$$w > f, \quad T\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - p = 0 \qquad \text{in the non-contact set,} \tag{7.11b}$$

with continuity of w and $\mathrm{d}w/\mathrm{d}x$ at the boundaries between the contact and non-contact sets. The two conditions (7.11) may usefully be combined to

$$(w - f)\left(T\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - p\right) = 0, \qquad \text{with} \quad w \geqslant f, \quad T\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - p \leqslant 0, \tag{7.12}$$

and this so-called *linear complementarity problem* can be written in another way which is very convenient computationally.

7.2 Variational formulation

We begin by defining a suitable function space in which we want the solution $w(x)$ to lie. To fix ideas we impose the simple boundary conditions $w(-1) = w(1) = 0$, although these are easily generalised. It is clear from solutions such as (7.5) that, although w and its first derivative are required to be continuous, the second derivative of w is usually discontinuous at the boundary of the contact set. We therefore seek w in the space of continuously differentiable functions that satisfy the given boundary conditions and do not penetrate the obstable, that is

$$\mathcal{V} = \left\{v \in C^1[-1,1] : v \geqslant f \text{ on } [-1,1], v(-1) = v(1) = 0\right\}. \tag{7.13}$$

Now we reformulate (7.12) as a comparison principle between the solution $w(x)$ and all other elements of $\mathcal{V}$. By integrating (7.12) with respect to x we obtain

$$0 = \int_{-1}^{1} (w - f)\left(p - T\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)\mathrm{d}x \tag{7.14}$$

and the right-hand side can easily be rearranged to

$$0 = \int_{-1}^{1} T\frac{\mathrm{d}^2w}{\mathrm{d}x^2}(v - w)\,\mathrm{d}x + \int_{-1}^{1} (v - f)\left(p - T\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)\mathrm{d}x + \int_{-1}^{1} p(w - v)\,\mathrm{d}x, \tag{7.15}$$

where $v(x)$ is any other element of the function space $\mathcal{V}$. Since $\mathrm{d}w/\mathrm{d}x$ is continuous, the first term may be integrated by parts, using the boundary conditions satisfied by v and w, to obtain

$$\int_{-1}^{1} T\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x}-\frac{\mathrm{d}w}{\mathrm{d}x}\right)\mathrm{d}x-\int_{-1}^{1} p(w-v)\,\mathrm{d}x=\int_{-1}^{1}(v-f)\left(p-T\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)\mathrm{d}x. \tag{7.16}$$

Then, using (7.12) and the condition $v \geqslant f$, we deduce that w satisfies *variational inequality*

$$T\int_{-1}^{1}\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x}-\frac{\mathrm{d}w}{\mathrm{d}x}\right)\mathrm{d}x \geqslant \int_{-1}^{1} p(w-v)\,\mathrm{d}x \quad \text{for all} \quad v\in\mathcal{V}. \tag{7.17}$$

It may be shown that the implication also follows in the other direction, so that (7.17) and (7.12) are equivalent. It may also be shown that w satisfies (7.17) if and only if it is the minimiser of the functional

$$U[v]=\int_{-1}^{1}\left\{\frac{T}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2+pv\right\}\mathrm{d}x \tag{7.18}$$

over all $v \in \mathcal{V}$. This has the obvious physical interpretation of minimising the elastic energy minus the work done by pressure over the virtual displacements v which prevent interpenetration of the string and the obstacle. This variational formulation is particularly useful when solving such problems numerically, and is ideally suited to the *finite element method*.

7.3 Other thin solids

It is straightforward to extend the model derived above to describe smooth contact between an elastic beam and a rigid substrate. In the non-contact set, the displacement w of the beam is governed by the equation

$$T\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4}-p=0 \tag{7.19}$$

where, as in Section 6, EI denotes the bending stiffness. Evidently (7.19) reduces to (7.1) as the bending stiffness tends to zero. In the contact set, (7.19) is simply replaced by the condition $w = f$.

Since (7.19) is fourth-order, we expect to need additional boundary conditions to specify a unique solution. Now, by balancing forces and moments at the boundary between the contact and non-contact sets, we deduce that w, $\mathrm{d}w/\mathrm{d}x$ and $\mathrm{d}^2w/\mathrm{d}x^2$ must all be continuous there.

We can obtain a variational formulation of the contact problem for a beam by incorporating the elastic bending energy into the integrand of (7.18), that is

$$\min_{w\geqslant f}\int_{-1}^{1}\left\{\frac{T}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2+\frac{EI}{2}\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2+pw\right\}\mathrm{d}x. \tag{7.20}$$

It is easy to show that (7.19) is the Euler–Lagrange equation associated with (7.20) in the absence of any contact.

To generalise the above theories into three dimensions, we now consider an elastic membrane with transverse displacement $w(x, y)$ making contact with a smooth obstacle $z = f(x, y)$

under an applied body force $p(x, y)$. Clearly the two-dimensional analogue of (7.1) is Poisson's equation

$$T\nabla^2 w = p(x, y), \tag{7.21}$$

where T is the uniform tension in the membrane. A force balance at the boundary of the contact set now reveals that w and its normal derivative must be continuous there, that is

$$w = f, \quad \frac{\partial w}{\partial n} = \frac{\partial f}{\partial n} \qquad \text{on the boundary of the contact set.} \tag{7.22}$$

In other words, w and $\boldsymbol{\nabla} w$ must be continuous everywhere.

The additional dimension makes analytical progress difficult except in simple cases such as axisymmetric problems. However, the numerical solution is straightforward in principle using a variational formulation analogous to (7.18), that is

$$\min_{w \geqslant f} \iint_D \left(\frac{T}{2} |\boldsymbol{\nabla} w|^2 + pw \right) \mathrm{d}x\mathrm{d}y, \tag{7.23}$$

and we can additionally incorporate bending stiffness as follows:

$$\min_{w \geqslant f} \iint_D \left(\frac{T}{2} |\boldsymbol{\nabla} w|^2 + \frac{EI}{2} \left(\nabla^2 w\right)^2 + pw \right) \mathrm{d}x\mathrm{d}y. \tag{7.24}$$

Example: Indentation of a taut membrane

Thin elastic objects are often subject to an in-plane tension, T. This tension can be important in numerous applications from Graphene (a one-molecule thick layer of carbon) to biological samples such as skin. A common way to measure this tension is to indent the object using, for example, an atomic force microscope, and measure the applied force, F, as a function of the imposed indentation depth, δ. Here, we consider a simple example to show the principles at work in such cases.

We consider axisymmetric deflections of a membrane, $w(r)$, caused by indentation of depth δ (measured relative to the boundaries); the indentation is imposed at the centre of the membrane, $r = 0$. Out of contact, small vertical deflections satisfy

$$T \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}w}{\mathrm{d}r} \right) = 0. \tag{7.25}$$

Assuming that $w(L) = 0$ (where the edge of the membrane is positioned at $r = L$) we therefore have that outside the contact region,

$$w(r) = A \log(r/L). \tag{7.26}$$

The behaviour in the contact region will depend on the shape of the indenter. For simplicity, we take a conical indenter, with shape $z = f(r) = -\delta + r/\tan\theta$, where θ is the angle of the cone. The shape of the membrane in the contact region, which we denote $0 \leqslant r < s$, is therefore

$$w(r) = -\delta + r/\tan\theta. \tag{7.27}$$

To determine the unknown constant A and the extent of the contact region, s, we must match the deflection and slope at the edge of the contact region. This gives

$$-\delta + s/\tan\theta = A\log(s/L), \quad \cot\theta = A/s, \tag{7.28}$$

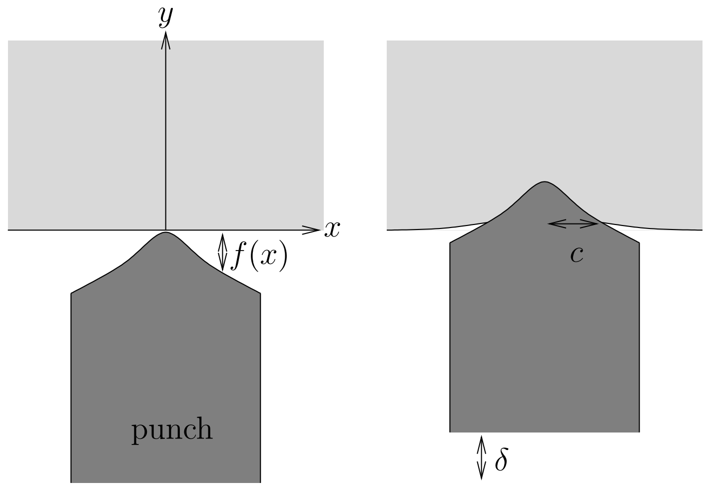

Figure 7.4: Definition sketch for contact between a rigid body and an elastic half-space.

from which we obtain

$$A = s \cot \theta \tag{7.29}$$

and

$$\delta = s \cot \theta \log(eL/s). \tag{7.30}$$

While it is not possible to determine s explicitly in terms of δ, this is enough for our purposes. In particular, to calculate the force F applied, we integrate the normal force, $N = -T\nabla^2 w$, over the contact surface

$$F = 2\pi \int_0^s r(T\nabla^2 w)\ \mathrm{d}r = 2\pi T s \cot \theta \tag{7.31}$$

so that $s = F \tan \theta / (2\pi T)$ and hence

$$\delta = \frac{F}{2\pi T} \log \left(\frac{2\pi e T L}{F \tan \theta} \right). \tag{7.32}$$

We note that the relationship between force and displacement is nonlinear. As such, inference of the tension T from measurements of indentation depth and force is actually a slightly subtle business. Finally, we note that the above analysis was predicated on the assumption of small membrane slopes, and so only holds for $\tan \theta \gg 1$.

7.4 Smooth contact in plane strain

As an intermediate case between the thin contact problems described above and general three-dimensional elastic contact, we will now consider contact in plane strain. For simplicity, we limit our attention to the problem of a rigid body, known as a *punch*, pushed a distance δ into the elastic half-space $y > 0$, as shown in Figure 7.4.

Let us suppose the geometry is symmetric about $x = 0$ so that the contact set is $-c < x < c$ for some positive c. Inside this region, the normal displacement is given in terms of the shape

$f(x)$ of the punch and, assuming smooth contact, the tangential stress is zero. In addition, there must be a positive reaction force, so that

$$v = \delta - f(x), \quad \tau_{xy} = 0, \quad \tau_{yy} < 0 \qquad \text{on} \quad y = 0, \ |x| < c. \tag{7.33a}$$

Outside the contact set, the surface traction is zero and there must be no inter-penetration, that is

$$\tau_{xy} = \tau_{yy} = 0, \quad v > \delta - f(x) \qquad \text{on} \quad y = 0, \ |x| > c. \tag{7.33b}$$

For general punch profile $f(x)$, this problem must be solved numerically. The variational formulations given above for thin solids suggest that this could be achieved by minimising the net elastic energy over all configurations that do not involve inter-penetration, that is

$$\min_{v(x,0) \geqslant \delta - f(x)} \int_0^\infty \int_{-\infty}^\infty \mathcal{W} \, \mathrm{d}x \mathrm{d}y, \tag{7.34}$$

where $\mathcal{W}$ is the strain energy density. It can be shown that this is indeed equivalent to solving the plane strain equations subject to the boundary conditions (7.33).