Problem Sheet 1

1. Assume that the stress and strain tensors in a linear isotropic solid are related by

$$\tau_{ij} = 2\mu e_{ij} + \lambda(e_{kk})\delta_{ij},$$

where λ and μ are constants (called the *Lamé constants*).

Find τ_{ij} when $u = \alpha x$, $v = -\beta y$, $w = -\beta z$, corresponding to uniaxial stretching of a bar. If the edge of the bar is traction-free, show that $\beta/\alpha = \lambda/2(\lambda + \mu) = \nu$, say (this is called *Poisson's ratio*). Based on your everyday experience, do you expect ν to be positive or negative?

Show that the the ratio of axial stress T to strain α is given by $E = \mu\,(3\lambda + 2\mu)\,/(\lambda + \mu)$ (this is called *Young's modulus*). Show that

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu = \frac{E}{2(1+\nu)}.$$

2. Starting from the unsteady Cauchy Momentum Equation including a body force $\boldsymbol{g}$ per unit mass, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \left\{\frac{1}{2}\rho\left|\frac{\partial u}{\partial t}\right|^2 + \mathcal{W}\right\}\mathrm{d}V = \iint_{\partial V}\frac{\partial \boldsymbol{u}}{\partial t}\cdot(\mathcal{T}\boldsymbol{n})\,\mathrm{d}S + \iiint_V \rho\boldsymbol{g}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}V \qquad (*)$$

for any volume V, where

$$\mathcal{W}(e_{ij}) = \frac{1}{2}e_{ij}\tau_{ij} = \frac{1}{2}\lambda\,(e_{kk})^2 + \mu e_{ij}e_{ij} = \frac{1}{2}\lambda\big(\mathrm{Tr}(\mathcal{E})\big)^2 + \mu\,\mathrm{Tr}\left(\mathcal{E}^2\right). \qquad (\dagger)$$

Interpret the terms in $(*)$ physically in terms of energy.

3. Show that $(\dagger)$ may be rearranged to

$$\mathcal{W}(e_{ij}) = \left(\frac{\lambda}{2}+\frac{\mu}{3}\right)(e_{kk})^2 + \mu\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right).$$

Deduce that the necessary and sufficient conditions for $\mathcal{W}(e_{ij})$ to have a global minimum at $e_{ij} = 0$ are $\mu > 0$ and $\lambda + 2\mu/3 > 0$.

4. Suppose the displacement $\boldsymbol{u}$ is specified on the boundary of an elastic body B. Use the calculus of variations[1] to show that, if $\boldsymbol{u}$ is chosen to minimise the integral

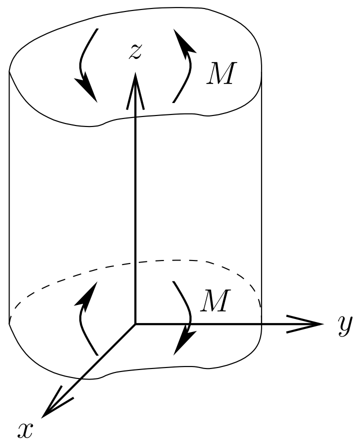
$$U = \iiint_B \{\mathcal{W}(e_{ij}) - \rho\boldsymbol{g}\cdot\boldsymbol{u}\}\,\mathrm{d}V,$$

then it satisfies the steady Navier equation.

If $\boldsymbol{u}$ is unspecified on ∂B, show that minimisation of U leads also to the *natural boundary condition* $\mathcal{T}\boldsymbol{n} = \boldsymbol{0}$.

[1] see for example Chapter 2 of F. B. Hildebrand 1965 *Methods of Applied Mathematics* (Dover)

5. Consider the *torsion* of a bar subject to a moment M.



Show that a displacement of the form

$$\boldsymbol{u} = \Omega\big(-yz, xz, \psi(x,y)\big)^{\mathrm{T}}$$

satisfies the steady Navier equation provided $\nabla^2\psi = 0$. Show also that the traction on the curved boundary of the bar is zero if

$$\frac{\partial\psi}{\partial n} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(x^2+y^2\right),$$

where s is arc-length along this boundary.

Suppose that the bar has flat ends at $z = 0$, $z = L$. Show that the torque exerted on each end is given by

$$M = \iint_D \left(x\tau_{yz} - y\tau_{xz}\right)\,\mathrm{d}x\mathrm{d}y = R\Omega,$$

where $D \subset \mathbb{R}^2$ is the cross-section of the bar and the *torsional rigidity* R is given by

$$R = \mu\iint_D \left\{x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x} + \left(x^2+y^2\right)\right\}\,\mathrm{d}x\mathrm{d}y.$$

For the case of a circular bar of radius a, evaluate ψ and hence show that

$$R = \frac{\pi a^4\mu}{2}. \tag{‡}$$

Explain why there exists a *stress function* $\phi(x,y)$ such that

$$\tau_{xz} = \mu\Omega\frac{\partial\phi}{\partial y}, \qquad \tau_{yz} = -\mu\Omega\frac{\partial\phi}{\partial x}.$$

Show that ϕ satisfies *Poisson's equation* $\nabla^2\phi = -2$ in D and that ϕ is constant on ∂D. Explain why this constant may be set to zero without loss of generality (this is called *choosing a gauge*), and show that, in this case,

$$R = 2\mu\iint_D \phi\,\mathrm{d}x\mathrm{d}y.$$

For a circular bar, evaluate ϕ and hence reproduce (‡).

6. Suppose the bar in Question 5 is hollow (as usually happens in practice) with inner and outer boundaries given by ∂D_i and ∂D_o respectively. Explain why in this case the boundary conditions for ϕ are $\phi = 0$ on ∂D_o and $\phi = k$ on ∂D_i, where k is constant, and show that the torsional rigidity is now given by

$$R = 2\mu \iint_D \phi \, \mathrm{d}x\mathrm{d}y + 2\mu k A,$$

where A is the area of the hole. Show also that k must be chosen so that ϕ satisfies

$$\oint_{\partial D_i} \frac{\partial \phi}{\partial n} \, \mathrm{d}s = -2A.$$

Hence evaluate ϕ when D is the circular annulus $a < r < b$ and show that the corresponding torsional rigidity is $R = \frac{\pi}{2}\mu\,(b^4 - a^4)$.

Reproduce this result using ψ instead of ϕ.