Problem Sheet 2

1. In plane strain, show that a smooth single-valued displacement can exist only if the the strain components e_{xx}, e_{xy} and e_{yy} satisfy the *compatibility condition*

$$\frac{\partial^2 e_{yy}}{\partial x^2} - 2\frac{\partial^2 e_{xy}}{\partial x \partial y} + \frac{\partial^2 e_{xx}}{\partial y^2} = 0.$$

Reformulate this relation in terms of the stress components τ_{xx}, τ_{xy} and τ_{yy}.

How many compatibility conditions do you think there are in three dimensions?

2. In the absence of a body force, the steady Navier equation takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} = 0, \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{r\theta}\right) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\tau_{r\theta}}{r} = 0,$$

in plane polar coordinates. Show that these are satisfied identically by introducing an Airy stress function $\mathfrak{A}$ such that

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \mathfrak{A}}{\partial \theta^2} + \frac{1}{r}\frac{\partial \mathfrak{A}}{\partial r}. \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \mathfrak{A}}{\partial \theta}\right), \qquad \tau_{\theta\theta} = \frac{\partial^2 \mathfrak{A}}{\partial r^2}.$$

[*These may alternatively be obtained by transforming the Cartesian relationships using the chain rule.*]

3. In plane strain, the two-dimensional stress tensor takes the form

$$\mathcal{T} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$$

with respect to *principal axes*, where τ_1 and τ_2 of are the *principal stresses*. Show that, if the axes are rotated through an angle θ, then $\mathcal{T}$ is transformed to

$$\mathcal{T}' = \begin{pmatrix} \tau_1 \cos^2\theta + \tau_2 \sin^2\theta & (\tau_2 - \tau_1)\sin\theta\cos\theta \\ (\tau_2 - \tau_1)\sin\theta\cos\theta & \tau_1 \sin^2\theta + \tau_2\cos^2\theta \end{pmatrix}.$$

Deduce that the maximum shear stress is $S = |\tau_1 - \tau_2|/2$.

Show that, with respect to arbitrary axes, S is given by

$$S^2 = \frac{\left(\tau_{xx} - \tau_{yy}\right)^2}{4} + \tau_{xy}^2.$$

[*The* Tresca yield criterion *states that a solid material will fail if S exceeds some critical* yield stress τ_Y.]

4. A gun barrel occupies the region $a < r < b$ in plane polar coordinates. A uniform pressure P is applied to the inner surface $r = a$ while the outer surface $r = b$ is traction-free. Assume that the displacement is purely radial, so that $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r$. By solving the Navier equation in polar coordinates, obtain the solution

$$u_r(r) = \frac{Pa^2}{2\left(b^2 - a^2\right)}\left(\frac{r}{\lambda + \mu} + \frac{b^2}{\mu r}\right),$$

and hence show that the maximum shear stress defined in Question 3 is given by

$$S = \frac{\tau_{\theta\theta} - \tau_{rr}}{2} = \frac{Pa^2b^2}{\left(b^2 - a^2\right)r^2}.$$

Deduce that the barrel will explode if

$$P > \tau_Y\left(1 - \frac{a^2}{b^2}\right),$$

where τ_Y is the Tresca yield stress.

5. Seek harmonic wave solutions $\boldsymbol{u} = \boldsymbol{a}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$ (real part assumed) of the dynamic Navier equation.

   Show that there exists a unique scalar A and vector $\boldsymbol{B}$ such that $\boldsymbol{a} = A\boldsymbol{k} + \boldsymbol{B}\times\boldsymbol{k}$ and $\boldsymbol{k}\cdot\boldsymbol{B} = 0$.

   Deduce that either $\boldsymbol{B} = \boldsymbol{0}$, $\rho\omega^2 = (\lambda + 2\mu)\left|\boldsymbol{k}\right|^2$ or $A = 0$, $\rho\omega^2 = \mu\left|\boldsymbol{k}\right|^2$.

   Show that the wave-speeds $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$ satisfy $c_p > c_s$.

6. An elastic medium occupies the half-space $y < 0$ and the surface $y = 0$ is stress-free. If the displacement is two-dimensional, with $\boldsymbol{u} = \big(u(x,y,t), v(x,y,t), 0\big)^{\mathrm{T}}$, obtain the boundary conditions

$$\left(c_p^2 - 2c_s^2\right)\frac{\partial u}{\partial x} + c_p^2\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \qquad \text{on} \quad y = 0.$$

   Show that a *Rayleigh wave* can propagate close to the surface, with

$$\boldsymbol{u} = \left(\boldsymbol{u}_p\mathrm{e}^{\kappa_p y} + \boldsymbol{u}_s\mathrm{e}^{\kappa_s y}\right)\exp\big\{\mathrm{i}\left(kx - \omega t\right)\big\},$$

   where $\kappa_p^2 = k^2 - \omega^2/c_p^2$ and $\kappa_s^2 = k^2 - \omega^2/c_s^2$. What restriction on the propagation speed $c = \omega/k$ will ensure that κ_p and κ_s are both real (and positive)?

   Deduce that the propagation c satisfies the equation

$$\left(2 - \frac{c^2}{c_s^2}\right)^2 = 4\left(1 - \frac{c^2}{c_p^2}\right)^{1/2}\left(1 - \frac{c^2}{c_s^2}\right)^{1/2},$$

   and confirm graphically that this has only one real root in the range $0 < c < c_s$.

7. A uniform beam of line density ϱ and length L lying along the x-axis under a tension T undergoes a small transverse displacement $w(x,t)\boldsymbol{k}$. Derive the governing equations

$$\frac{\partial T}{\partial x} = 0, \qquad\qquad T\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 M}{\partial x^2} - \varrho g = \varrho\frac{\partial^2 w}{\partial t^2},$$

where M is the clockwise bending moment exerted on each cross-section of the beam and the gravitational acceleration is $\boldsymbol{g} = -g\boldsymbol{k}$.

Use an exact solution of the steady Navier equation to justify the constitutive relation

$$M = -EI\frac{\partial^2 w}{\partial x^2},$$

where E is Young's modulus and I is the moment of inertia of the cross-section about the y-axis.

If gravity is negligible and no transverse force is applied at the ends, which are clamped horizontally, justify the boundary conditions $\partial w/\partial x = \partial^3 w/\partial x^3 = 0$ at $x = 0$ and $x = L$. Show that the natural frequencies ω of the beam are given by

$$\omega^2 = \frac{n^2\pi^2}{\varrho L^2}\left(\frac{n^2\pi^2 EI}{L^2} + T\right),$$

and deduce that the beam is unstable if $T < -\pi^2 EI/L^2$.