## Problem Sheet 2

QUESTION 1. **Null Lagrangian.**

(1) Give two examples of a Null-Lagrangian $L(\nabla u, u, x)$ (and explain in particular why the functions you propose are Null-Lagrangians.)

(2) Define for real $n \times n$ matrices $P \in \mathbb{R}^{n\times n}$ the map

$$L(P) = \mathrm{tr}(P^2) - (\mathrm{tr}(P))^2 .$$

where $\mathrm{tr}(P)$ denotes the trace of the matrix $P$. Show that $L$ is a Null-Lagrangian.

QUESTION 2. **Equivalence between Retraction Principle and Brouwer's FPT.** Let $B$ be the closed unit ball in $\mathbb{R}^n$. Using Brouwer's Fixed Point Theorem, show that there does not exist a retraction $r$ from $B$ to $\partial B$, i.e. a map $r : B \to \partial B$ such that $r$ restricted to $\partial B$ is the identity map.

Hint: by contradiction, consider the map $g(x) = -r(x)$.

QUESTION 3. **Leray-Schauder/Schaefer Theorem.**

- Prove the following result Let $X$ be a Banach space and $T\colon X \to X$ be a compact map with the following property: there exists $R > 0$ such that the statement ($x = \tau T x$ with $\tau \in [0, 1)$) implies $\|x\|_X < R$. Then $T$ has a fixed point $x^*$ such that $\|x^*\|_X \le R$.

  Hint: Consider the operators

$$T_n(x) := \begin{cases} Tx & \text{if } \|Tx\|_X \le R + \frac{1}{n}, \\ \frac{R+1/n}{\|Tx\|_X} Tx & \text{else} \end{cases}$$

  on a suitable domain and prove that they are compact.

- Let $T : X \to X$ a compact map such that there exists $R > 0$ such that $\|Tx - x\|_X^2 \ge \|Tx\|_X^2 - \|x\|_X^2$ when $\|x\|_X \ge R$. Show that $T$ admits a fixed point.

QUESTION 4. **Leray's eigenvalue problem**. Let $K : [a, b] \times [a, b] \to (0, \infty)$ be a continuous and positive function and consider the integral operator $T : C^0([a, b]) \to C^0([a, b])$ defined by

$$(Tu)(x) = \int_a^b K(x, t)u(t)\, dt.$$

Prove that $T$ has at least one non-negative eigenvalue $\lambda$ whose eigenvector is a continuous non-negative function $u$, i.e. there exist $\lambda \ge 0$ and a non-negative $u$ so that

$$\int_a^b K(x, t)u(t)\, dt = \lambda u(x).$$

*Hint: consider, on an appropriate closed convex set $M$, the function*

$$F(u) = \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$

*and apply one of the versions of Schauder's Fixed Point Theorem with the help of Arzéla-Ascoli Theorem. To find a suitable set $M$ think about what property all functions $F(u)$ have in common.*

QUESTION 5. **Integral operators on $L^2(\Omega)$ vs. $C(\bar{\Omega})$** As always, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain.

- Let $a : \bar{\Omega} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous map, and let
$$A(u)(x) = \int_\Omega a(x, y, u(y))dy.$$
show that $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined and compact. (Hint: use Arzela-Ascoli Theorem).
- Let $k \in L^2(\Omega \times \Omega)$ and define
$$(Ku)(x) = \int_\Omega k(x, y)u(y)dy.$$
Show that $K : L^2(\Omega) \to L^2(\Omega)$ is well defined and compact. You can use for example that $C_0^\infty(\Omega \times \Omega)$ is dense in $L^2(\Omega \times \Omega)$, and therefore there is a sequence $k_m \in C_0^\infty(\Omega \times \Omega)$ such that $k_m \to k$ in $L^2(\Omega)$.
- Give an example of continuous $a$ such that $A$ (defined as above) is not well defined as an operator from $L^2(\Omega) \to L^2(\Omega)$.

QUESTION 6. **Continuous maps.** Let $g \in C(\mathbb{R} \times \mathbb{R}^n)$ be such that $g(z, p) \leq a + b|z|^\alpha + c|p|$, where $a$,$b$ and $c$ are non negative constants, and $2\alpha < 2^*$, where $2^* = 2n/(n-2)$ if $n \geq 3$, and $2* = \infty$ if $n = 1, 2$. Then the map $u \mapsto g(u, \nabla u)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and maps bounded subsets of $H_0^1(\Omega)$ to bounded subsets of $L^2(\Omega)$.

*Hint: rewrite* $g(u, \nabla u) = \tilde{g}(u, \frac{\nabla u}{|\nabla u|^\nu})$ *for a suitable function* $\tilde{f}$ *and a suitable exponent* $0 < \nu < 1$ *and apply Lemma 25 from the lecture*