Exercise 1

Exercise 1. Use the method of characteristics to solve the problem

$$u_y + (2x+u)u_x - (x+2u) = 0, \quad u(x,0) = 1-x.$$

Exercise 2. Use the method of characteristics to find all possible solutions of the problem

$$u_x^2 + u_y^2 = 1, \qquad u = 0 \text{ on } x^2+y^2 = 1.$$

Exercise 3. Consider the Burgers equation

$$u_t + (u^2/2)_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty).$$

(i) If u is a C^1 solution defined for all $t > 0$, show that for each fixed $t > 0$, the function $x \to u(x,t)$ is monotonically increasing.
(ii) If $u(x,0) = u_0(x)$ is not nondecreasing, then the Burgers equation can not have a C^1 solution defined for all $t > 0$.

Exercise 4. Consider the equation of conservation laws

$$u_t + f(u)_x = 0, \tag{0.1}$$

where f is a C^2 function.

(i) Show that if u is a C^1 solution of (0.1), then $v := f'(u)$ is a C^1 solution of the Burgers equation $v_t + (v^2/2)_x = 0$.
(ii) Can the discontinuous solution of (0.1) be always mapped into a weak solution of the Burgers equation? Why?

Exercise 5. Consider the conservation law with viscous term

$$u_t + f(u)_x = \varepsilon u_{xx},$$

where $f \in C^2(\mathbb{R})$ and $\varepsilon > 0$ is a small number. Assume that there is a traveling wave solution of the form $u(x,t) = w(\frac{x-st}{\varepsilon})$ for some number s and some C^2 function $w : \mathbb{R} \to \mathbb{R}$. Assume also that $w(-\infty) = u_l$ and $w(+\infty) = u_r$ with $u_l \neq u_r$. Show that

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \quad \text{and} \quad f'(u_l) \geq s \geq f'(u_r).$$

(These resemble the Rankine-Hugoniot condition and the Lax entropy condition)

Exercise 6. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\varphi \geq 0, \quad \int_{\mathbb{R}^n} \varphi(x)dx = 1, \quad \text{supp}(\varphi) \subset \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

For any $\varepsilon > 0$ set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. For any locally integrable function f on $\mathbb{R}^n$ we define its convolution with φ_ε as

$$(f * \varphi_\varepsilon)(x) := \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x-y)dy, \quad x \in \mathbb{R}^n.$$

Show that

(i) $f * \varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$ for any $\varepsilon > 0$;
(ii) If $f \in C(\mathbb{R}^n)$, then $f * \varphi_\varepsilon$ uniformly converges to f on any compact subset of $\mathbb{R}^n$ as $\varepsilon \to 0$;
(iii) If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then $\|f * \varphi_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$ and $\|f * \varphi_\varepsilon - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$.
(Hint: You may use the fact that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.)