Exercise 2

Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be a function defined on a finite interval $[a, b]$. The total variation of f on $[a, b]$ is defined by

$$TV(f; a, b) := \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken with respect to all possible partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of $[a, b]$. If $TV(f; a, b)$ is finite, we say f is of bounded variation on $[a, b]$.

(i) Show that the function

$$f(x) = \begin{cases} x \sin(\pi/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is not of bounded variation on $[0, 1]$.

(ii) Show that the function

$$f(x) = \begin{cases} x^2 \sin(\pi/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is of bounded variation on $[0, 1]$.

(iii) If $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous on $[a, b]$, then f is of bounded variation on $[a, b]$ with

$$TV(f; a, b) = \int_a^b |f'(x)| dx.$$

(Hint: You may use the fact that if F is nondecreasing on $[a, b]$ then $\int_a^b F'(t)dt \leq F(b) - F(a)$.)

Exercise 2. For the Burgers equation $u_t + (u^2/2)_x = 0$ applied to each of the sets of initial data $u(x, 0) = u_0(x)$ in the following, determine the exact solution for all $t > 0$.

(i)

$$u_0(x) = \begin{cases} 1 & x < -1, \\ 0 & -1 < x < 1 \\ -1 & x > 1; \end{cases}$$

(ii)

$$u_0(x) = \begin{cases} -1 & x < -1, \\ 0 & -1 < x < 1 \\ 1 & x > 1 \end{cases}$$

Exercise 3. Compute explicitly the unique entropy solution of

$$\begin{aligned} u_t + (u^2/2)_x = 0 \quad & \text{in } \mathbb{R} \times (0, \infty), \\ u = g \quad & \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

for

$$g(x) = \begin{cases} 2 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Exercise 4. (i) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function whose gradient is denoted by ∇f. Then the following three statements are equivalent:

(a) f is convex;

(b) $f(x_1) \geq f(x_0) + \langle \nabla f(x_0), x_1 - x_0 \rangle$ for all $x_0, x_1 \in \mathbb{R}$;
(c) $\langle \nabla f(x_1) - \nabla f(x_0), x_1 - x_0 \rangle \geq 0$ for all $x_0, x_1 \in \mathbb{R}$.

where, for any $x, y \in \mathbb{R}^n$, $\langle x, y \rangle$ denotes their inner product.

(ii) Let $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 convex function. Then f achieves its minimum at $x_0 \in \mathbb{R}^n$ if and only if $\nabla f(x_0) = 0$.

(iii) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. We use f^* to denote its Legendre-Fenchel conjugate, i.e.

$$f^*(\xi) = \sup_{x \in \mathbb{R}^n} \{ \langle \xi, x \rangle - f(x) \}, \quad \xi \in \mathbb{R}^n.$$

(a) For $f(x) = |x|^p/p$ for $x \in \mathbb{R}$ with $p > 1$, calculate f^* and use the result to show the Young's inequality

$$\xi x \leq \frac{1}{p}|x|^p + \frac{1}{q}|\xi|^q$$

for any $x, \xi \in \mathbb{R}$ and $p, q > 1$ with $1/p + 1/q = 1$.

(b) Let A be an $n \times n$ symmetric positive definite matrix and let $b \in \mathbb{R}^n$. Consider the function $f(x) := \langle x, Ax \rangle + \langle b, x \rangle$. Determine f^*.

Exercise 5. Consider the initial value problem of Burgers equation

$$\begin{cases} u_t + (u^2/2)_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $u_0 \in L^\infty(\mathbb{R})$. It is known that this problem may not have a unique weak solution. One way to pick out the unique physically correct solution is to consider the solutions of viscous Burgers equation

$$\begin{cases} u_t + (u^2/2)_x = \varepsilon u_{xx} & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{0.1}$$

for small $\varepsilon > 0$ and then investigate the limit as $\varepsilon \to 0$.

(i) Let $h(x) = \int_0^x u_0(y)dy$. If w is the unique smooth solution of the problem

$$\begin{cases} w_t + \frac{1}{2}w_x^2 = \varepsilon w_{xx} & \text{in } \mathbb{R} \times (0, \infty), \\ w(x,0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{0.2}$$

then $u := w_x$ is the solution of the problem (0.1).

(ii) Let w be a smooth function satisfying $w_t + \frac{1}{2}w_x^2 = \varepsilon w_{xx}$. Find a nonconstant function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $z := \varphi(w)$ solves the heat equation $z_t = \varepsilon z_{xx}$.

(iii) Recall that the solution of the initial value problem of heat equation

$$\begin{cases} z_t = \varepsilon z_{xx} & \text{in } \mathbb{R} \times (0, \infty), \\ z = z_0 & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

is given by

$$z(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} z_0(y) \exp\left(-\frac{(x-y)^2}{4\varepsilon t} \right) dy.$$

Let u^ε denote the solution of (0.1). Use the above formula and the information from (i) and (ii) to find the explicit formula of u^ε.

(iv) Before discussing the limit of u^ε as $\varepsilon \to 0$, consider the limit

$$A = \lim_{\varepsilon \to 0} \frac{\int_{-\infty}^{\infty} \ell(y) \exp(-\kappa(y)/\varepsilon) dy}{\int_{-\infty}^{\infty} \exp(-\kappa(y)/\varepsilon) dy},$$

where $\ell : \mathbb{R} \to \mathbb{R}$ are continuous functions. Assume that κ is locally Lipschitz (i.e. Lipschitz on any compact subset of $\mathbb{R}$), $\kappa(y) \geq c_0|y|^2 - c_1$ and $|\ell(y)| \leq c_2 + c_3|y|^p$ with $p \geq 0$ for some positive constants c_0, c_1, c_2 and c_3. Assume also that there exists a unique $y_0 \in \mathbb{R}$ such that $\kappa(y_0) = \min_{y\in\mathbb{R}} \kappa(y)$. Then the limit A exists and $A = \ell(y_0)$.

(v) Applying (iv) to show that

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = \frac{x - y(x,t)}{t}$$

for those $(x,t) \in \mathbb{R} \times (0,\infty)$ such that

$$\min_{y\in\mathbb{R}} \left\{ \frac{(x-y)^2}{2t} + h(y) \right\}$$

has a unique minimizer, denoted by $y(x,t)$. Compare it with the result in Theorem 18 from lecture notes.