Exercise 3

Exercise 1. For $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, let

$$\mathcal{C}_{x_0,t_0} := \{(x,t) : 0 \le t \le t_0 \text{ and } |x - x_0| \le t_0 - t\}$$

be the backward light cone with vertex (x_0, t_0). Let $u \in C^2(\mathcal{C}_{x_0,t_0})$ satisfy

$$u_{tt} - \triangle u = F(u, \partial u) \quad \text{in } \mathcal{C}_{x_0,t_0},$$

where $F \in C^1(\mathbb{R})$ with $F(0,0) = 0$. If $u(x,0) = u_t(x,0) = 0$ for $|x - x_0| \le t_0$, then $u \equiv 0$ in $\mathcal{C}_{x_0,t_0}$.
(Hint: consider $E(t) = \int_{B_{t_0-t}(x_0)} (u(x,t)^2 + u_t(x,t)^2 + |\nabla u(x,t)|^2) dx$)

Exercise 2. (i) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. For a fixed $\omega \in \mathbb{S}^2$ and $0 < \epsilon < 2$, let

$$\Gamma_\epsilon := \{y \in \mathbb{S}^2 : y \cdot \omega \ge 1 - \epsilon\}.$$

Show that the area of Γ_ϵ is $2\pi\epsilon$.

(ii) Let $f, g \in C^\infty(\mathbb{R}^3)$ satisfy $f(x) = g(x) = 0$ for $|x| > R$, and let $u(t,x)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt} - \triangle u = 0 & \text{in } \mathbb{R}^3 \times [0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^3. \end{cases}$$

Show that $u(x,t) = 0$ if $|t - |x|| > R$ and satisfies the decay estimate

$$|u(t,x)| \le C(1+t)^{-1}, \quad \forall (x,t) \in \mathbb{R}^3 \times [0,\infty)$$

with a constant C depending only on R, $\|g\|_{L^\infty}$, $\|\nabla g\|_{L^\infty}$ and $\|h\|_{L^\infty}$. (Hint: use Huygens' principle and Kirchoff formula)

Exercise 3. (i) Let $u \in C^2(\mathbb{R}^n \times [0,T])$ be a classical solution of the Cauchy problem of wave equation

$$\begin{cases} \Box u := u_{tt} - \triangle u = f(x,t) & \text{in } \mathbb{R}^n \times (0,T], \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^n, \end{cases} \tag{0.1}$$

Show that for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))$ there holds

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} f\varphi dx dt = \int_0^T \int_{\mathbb{R}^n} u \Box \varphi dx dt + \int_{\mathbb{R}^n} g(x)\varphi_t(x,0) dx \\ - \int_{\mathbb{R}^n} h(x)\varphi(x,0) dx. \end{aligned} \tag{0.2}$$

(ii) Let $g, h \in L^1_{loc}(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n \times [0,T])$. A function $u \in L^1_{loc}(\mathbb{R}^n \times [0,T])$ is called a weak solution of (0.1) if (0.2) holds for all $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))$. Show that, for given $g \in C^2(\mathbb{R}^n)$, $h \in C^1(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T])$, if $u \in C^2(\mathbb{R}^n \times [0,T])$ is a weak solution, then u is also a classical solution.

Exercise 4. Let $g, h \in L^1_{loc}(\mathbb{R})$ and define

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y) dy.$$

Show that u is a weak solution of the Cauchy problem

$$\begin{cases} \Box u := u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}. \end{cases}$$

(Hint: consider the transformation $\xi = x + t$, $\eta = x - t$)

Exercise 5. (i) In the Minkowski space $(\mathbb{R}^{1+n}, \mathbf{m})$, let $\Box = \mathbf{m}^{\alpha\beta}\partial_\alpha\partial_\beta$, $L_0 = x^\rho\partial_\rho$ and $\Omega_{\mu\nu} = (\mathbf{m}^{\rho\mu}x^\nu - \mathbf{m}^{\rho\nu}x^\mu)\partial_\rho$. Show that

$$[\Omega_{\mu\nu}, L_0] = 0, \quad [\partial_\mu, L_0] = \partial_\mu, \quad [\Box, \Omega_{\mu\nu}] = 0, \quad [\Box, L_0] = 2\Box$$

where, for any two operators A and B, $[A, B] := AB - BA$ denotes their commutator.

(ii) In the Minkowski space $(\mathbb{R}^{1+3}, \mathbf{m})$, consider the vector field

$$X = (1 + t^2 + |x|^2)\partial_t + 2tx^i\partial_i.$$

Show by direct calculation that X is a conformal Killing vector field with ${}^{(X)}\pi = 4t\mathbf{m}$.