Exercise 4: Part I

Exercise 1. Define on $\mathbb{R}^{2+1}$ the functions,

$$u = t - r, \quad \underline{u} = t + r,$$
$$\rho = \sqrt{t^2 - r^2}, \text{ if } t \geq r$$

where $r := |x|$ and $\partial_r = \frac{\sum_i x^i \partial_i}{r}$.
(i) Show that if $r \neq 0$, u and $\underline{u}$ are solutions for the eikonal equation

$$(\partial_t \phi)^2 - \sum_{i=1}^{2} (\partial_i \phi)^2 = 0,$$

(ii) Show that ρ is the solution for

$$(\partial_t \phi)^2 - \sum_{i=1}^{2} (\partial_i \phi)^2 = 1$$

(iii) Let

$$L = \partial_t + \partial_r, \underline{L} = \partial_t - \partial_r,$$

show that

$$\mathbf{m}(L, L) = \mathbf{m}(\underline{L}, \underline{L}) = 0, \quad \mathbf{m}(L, \underline{L}) = -2$$

where $\mathbf{m}$ is the Minkowski metric.
(iv) Let $S = t\partial_t + \sum_{i=1}^{2} x^i \partial_i$. Represent S in terms of $L, \underline{L}$ and $u, \underline{u}$.
(v) Denote by $\mathbf{D}\rho$ the gradient of ρ in $\mathbb{R}^{2+1}$, express S in terms of $\mathbf{D}\rho$ in the region $t > |x|$.

Exercise 2. Consider the region $D = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}, |t| \geq |x|, t \geq 2\}$. Denote by Σ_t the level set of each fixed t in D. Let Ω be the vector field $x_1 \partial_2 - x_2 \partial_1$.

Let ϕ be the solution of $\Box \phi = 0$. We denote the standard energy by

$$E(t) = \frac{1}{2} \int_{\Sigma_t} \{(\partial_t \phi)^2 + \sum_{i=1}^{2} (\partial_i \phi)^2\} dx.$$

(i) Consider $\partial_t \phi \cdot \Box \phi$. By using divergence theorem in a suitable region, establish an energy identity for ϕ, i.e.

$$E(\tau) = E(2) + \text{Flux}, \quad \text{if } \tau > 2$$

where the flux is an integral on $\{t = |x|, 2 \leq t \leq \tau\}$ with positive integrand.

Represent the flux in terms of $L\phi$ and $\Omega\phi$.

(ii) Points to ponder:

Establish an energy identity in the region $\{\rho \leq \rho_0, t \leq \tau\} \subset D$. Here we assume $\tau > \rho_0$.

Hint: Prove that

$$2(\partial_t^2 - \sum_{i=1}^{2} \partial_{x_i}^2)\phi \partial_t \phi = \partial_t(|\boldsymbol{D}\phi|^2) - 2 \sum_{1}^{2} \partial_{x_j}(\partial_t \phi \partial_{x_j} \phi).$$

Then redefine energy on the upper boundary

$$\tilde{H}_{\rho_0} = \{(t, x), \sqrt{t^2 - r^2} = \rho_0, t \leq \tau\} \cup \{t = \tau, r \geq \sqrt{\tau^2 - \rho_0^2}\}$$

by finding the outward normal $\vec{n}$ on each part of the boundary.