Lie Groups

Section C course Hilary 2018

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Example sheet 2

1. The algebra of *quaternions* is defined as

$$\mathbb{H} = \{a + bi + cj + dk \ : \ a, b, c, d \in \mathbb{R}\}$$

where i, j, k satisfy the relations

$$ij = k = -ji \ : \ i^2 = j^2 = k^2 = -1$$

(i) Show that these relations imply $jk = i = -kj$ and $ki = j = -ik$.

(ii) Show that the algebra of quaternions may be identified with the algebra of matrices,

$$\left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \ : \ z, w \in \mathbb{C} \right\}.$$

(iii) If $q = a + bi + cj + dk \in \mathbb{H}$, we define the *quaternionic conjugate* to be

$$\bar{q} = a - bi - cj - dk$$

and the *norm* of q to be the nonnegative real number $|q|$ such that $|q|^2 = q\bar{q}$.

Show that $q\bar{q}$ is indeed real and nonnegative, so $|q|$ is well-defined. Deduce that $q \neq 0$ has a multiplicative inverse $q^{-1} = \frac{\bar{q}}{|q|^2}$.

Show also that

$$|q_1 q_2| = |q_1| \cdot |q_2| \quad : \quad |q^{-1}| = |q|^{-1}.$$

Viewing $\mathbb{H}$ as a real 4-dimensional vector space, check that $|q|$ is the usual norm on $\mathbb{R}^4$.

(iv) We define the Lie group

$$\mathrm{Sp}(n) = \{A \in GL(n, \mathbb{H}) : A^* A = Id_n\}$$

where A^* denotes the quaternionic conjugate transpose of A (ie the ij entry of A^* is the quaternionic conjugate of the ji entry of A).

Show that

$$\mathrm{Sp}(1) = \mathrm{SU}(2)$$

and hence that $\mathrm{Sp}(1)$ is topologically the 3-sphere.

For $q \in \mathbb{H} \setminus \{0\}$ define

$$\mathcal{A}_q : \mathbb{H} \to \mathbb{H}, \quad : \quad p \mapsto qpq^{-1}.$$

Show that $\mathcal{A}_q$ is an orthogonal map (viewing $\mathbb{H}$ as $\mathbb{R}^4$).

By considering the orthogonal complement of $\mathbb{R} = \mathbb{R} \cdot 1 \subset \mathbb{H}$, deduce that $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \subset \mathbb{H} \setminus \{0\}$ acts on $\mathbb{R}^3$ by rotations.

Explain briefly why this gives a homomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2) \to \mathrm{SO}(3)$ with kernel $\{\pm 1\}$.

2. Check these properties of $\exp : \mathrm{Lie}(G) \to G$.

(i) $\mathrm{Image}(\exp) \subset G_0 =$ connected component of $1 \in G$;
(ii) $\exp((t+s)v) = \exp(tv)\exp(sv)$ for all $t, s \in \mathbb{R}$;
(iii) $(\exp v)^{-1} = \exp(-v)$;
(iv) if $g = \exp(v)$ then it has an n-th root;
(v) the following map is not surjective

$$\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$$

3. Prove directly that **ad** is a Lie algebra homomorphism by using the fact that $\mathbf{ad}(X) \cdot Z = [X, Z]$.

Show that

$$v_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is a basis for $\mathfrak{so}(3) \subset \mathrm{Mat}_{3\times 3}(\mathbb{R})$.

By computing all brackets $[v_i, v_j]$, show that

$$\mathfrak{so}(3) \cong (\mathbb{R}^3, \text{cross product}), \quad : \quad v_i \mapsto \text{standard basis vector } e_i$$

is a Lie algebra isomorphism.

Via this isomorphism we identify $\mathrm{End}(\mathfrak{so}(3))$ with 3×3 matrices. Compute the matrices $\mathbf{ad}(v_i)$.

By computing $\langle v_i, v_j \rangle$ show that the **Killing form**

$$\langle v, w \rangle = \mathrm{Trace}(\mathbf{ad}(v)\mathbf{ad}(w)) \in \mathbb{R}$$

is a negative definite scalar product on $\mathfrak{so}(3)$.

4. Show that for a matrix group G, we have $\exp(gXg^{-1}) = g\exp(X)g^{-1}$ for all $g \in G$ and $X \in \mathfrak{g}$.

Consider the subgroup T of the unitary group $U(n)$ consisting of diagonal matrices. Show that T is a torus $T^n \cong (S^1)^n$ and that T lies in the image of the exponential map $\exp : \mathfrak{u}(n) \to U(n)$.

Deduce that $\exp : \mathfrak{u}(n) \to U(n)$ is surjective.