Lie Groups

Section C course Hilary 2018

dancer@maths.ox.ac.uk

Example sheet 1

1. Suppose G_1, G_2 are Lie groups. Show that $G_1 \times G_2$ is a Lie group in a natural way. Deduce that the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ is a Lie group.

Find a map $\pi : \mathbb{R}^n \to T^n$ that allows you to identify T^n with the quotient group $\mathbb{R}^n/\mathbb{Z}^n$. Which vector fields on $\mathbb{R}^n$ project under $d\pi$ to vector fields on T^n? Do all vector fields on T^n arise in this way?

Which vector fields X on T^n are *left-invariant*, that is, satisfy

$$(dL_g)_h(X|_h) = X_{gh}?$$

2. Use the implicit function/submanifold theorem to prove that the orthogonal group $O(n)$ is a Lie group. Compute the dimension of $O(n)$ and find the tangent space $T_I O(n)$. Show also that $O(n)$ is compact.

3. Show that the *tangent bundle* $TG = \bigsqcup_{g \in G} T_g G$ of a Lie group G is canonically identifiable with $G \times T_I G$. [*Hint. consider the left translation map* $L_g : G \to G$, $L_g(h) = gh$.]

Deduce that any Lie group of dimension n has n non-vanishing vector fields which are linearly independent at each point of G.

Show that the 3-dimensional sphere S^3 is a Lie group by identifying it with

$$SU(2) = \{2 \times 2 \text{ complex matrices } A \text{ with } A^* A = I,\ \det A = 1\}$$

where A^* denotes the conjugate transpose of A.

Show that the 2-dimensional sphere S^2 *cannot* be a Lie group. [*Hint. you may quote the "hairy ball theorem" from algebraic topology*]

4. Let $\varphi : M \to N$ be a *diffeomorphism* of manifolds (a smooth map with smooth inverse). For a vector field X on M define the *push-forward* vector field $Z = \varphi_* X$ on N by

$$Z|_y = d\varphi_x(X|_x)$$

where $x = \varphi^{-1}(y)$. Show that for any function $f : N \to \mathbb{R}$,

$$(\varphi_* X) \cdot f = (X \cdot (f \circ \varphi)) \circ \varphi^{-1}.$$

Deduce that $[\varphi_* X, \varphi_* Y] \cdot f = \varphi_* [X, Y] \cdot f$, and hence that

$$[\varphi_* X, \varphi_* Y] = \varphi_* [X, Y].$$

Let G be a Lie group. Prove the following characterization of left-invariant vector fields:

$$(L_g)_* X = X \quad \text{for all } g \in G,$$

Let $\mathrm{Lie}\, G$ denote the set of left-invariant vector fields. Deduce that, if $X, Y \in \mathrm{Lie}\, G$, then also $[X, Y] \in \mathrm{Lie}\, G$.

5. Let G be a Lie group, and let G_0 denote the connected pathcomponent of G containing the identity (we call G_0 the *identity component* of G).

Show that G_0 is a normal subgroup of G. If $G = O(n)$ what is G_0? Is it true in this example that $G \cong G_0 \times (G/G_0)$ as groups?