Geometric Group Theory

Problem Sheet 4

The starred exercises are optional.

1. Groups as geometric objects

1. Show that the Cayley graph Γ of an infinite finitely generated group G contains a bi-infinite geodesic.

2. i) Show that the relation of quasi-isometry of metric spaces $\sim$ is an equivalence relation.

ii) Let S_1, S_2 be finite generating sets of a group G. Show that $\Gamma(S_1, G) \sim \Gamma(S_2, G)$.

3. i) Show that any metric space X has a $(1,1)$-net.

ii) Show that if $N \subset X$ is a net then X is quasi-isometric to N.

iii) Show that X is quasi-isometric to Y if and only if there are nets $N_1 \subset X, N_2 \subset Y$ and a bilipschitz map $f : N_1 \to N_2$.

iv) Let G be a f.g. group. Show that $H < G$ is a net in G if and only if H is a finite index subgroup of G.

4. Show that the groups $\mathbb{Z}^2$ and $\mathbb{Z}^3$ are not quasi-isometric. (*hint*: growth)

5. Show that $\mathbb{F}_2 \times \mathbb{Z}$ has one end (where $\mathbb{F}_2$ is the free group of rank 2).

2. Hyperbolic spaces and Groups

6. Let X be a δ-hyperbolic geodesic metric space. If L is a geodesic in X and $a \in X$ we say that $b \in L$ is a projection of a to L if

$$d(a,b) = \inf\{d(a,x) : x \in L\}.$$

Show that if b_1, b_2 are projections of a to L then $d(b_1, b_2) \leq 2\delta$.

7. Let $G =< S|R >$ be a torsion free δ-hyperbolic group. Show that if $g^3 = h^3$ then $g = h$.

8. Let $G =< S|R >$ be δ-hyperbolic group. Show that G has no subgroup isomorphic to $< x, t | txt^{-1} = x^2 >$.

9. Show that a hyperbolic group has no subgroup isomorphic to $(\mathbb{Q}, +)$.

10. (*) Let $G =< S|R >$ be a Dehn presentation of a δ-hyperbolic group. Show that we can decide whether a word w on S represents an infinite order element.

11. (*) Let $G = < S|R >$ be a Dehn presentation of a δ-hyperbolic group. Show that we can decide whether a word w on S lies in the subgroup $< v >$.

3. Projects related to the course. Attempt at least two of these.

Project 1. We know that a free product of two residually finite groups is residually finite. The objective of the following is to show that the same is true for finite graphs of groups with residually finite vertex groups and finite edge groups.

i. Let $\rho_1, \rho_2 : G \to Symm(A)$ be two free actions of a finite group G on a finite set A. Show that there is some $e \in Symm(A)$ such that

$$e\rho_1(g)e^{-1} = \rho_2(g), \text{ for all } g \in G.$$

ii. Let $H = \pi_1(G, Y, a_0)$. Show that if Y is finite and all edge and vertex groups are finite then H has a finite index subgroup which is free. (*hint:* define an action of $F(G, Y)$ on the cartesian product of all vertex groups. Use part i) to make sure relations involving edges are satisfied).

iii. Suppose that $H = \pi_1(G, Y, a_0)$ where Y is finite all edge groups of Y are finite and all vertex groups of Y are residually finite. Show that H is residually finite. (*hint:* For each $g \neq 1$ map H to an appropriate fundamental graph of a graph of groups with *finite* vertex groups. Then use the previous exercise).

Project 2. Ends of Groups.

i. Show that if a finitely generated group G splits over a finite group then G has more than 1 end.

(*Hint*: This can be done either by constructing the Cayley graph Γ or by normal forms. If e.g. $G = A *_C B$ note that words of the form $(ab)^n$ and $(ba)^n$ lie in different components of $\Gamma \setminus C$.)

ii. Show that two quasi-isometric locally finite graphs have the same number of ends. Deduce that the number of ends of a finitely generated group is well defined (ie it does not depend on the Cayley graph that we pick).

iii. Show that a finitely generated group has 0,1,2 or ∞ ends.

Project 3. The objective of this project is to show that torsion free groups quasi-isometric to free groups are free.

Assume that a finitely generated group G is quasi-isometric to the free group F_n (with $n \geq 2$)

i. Show that G has infinitely many ends. (You may use the results of Project 2).

ii. Consider the Grusko decomposition of G as a free product: $G = G_1 * ... * G_k * F_s$. Show that none of the G_i's is 1-ended.

Hint: Note that if G_i is infinite then its Cayley graph contains a bi-infinite geodesic.

iii. Show that if H is a torsion free 2-ended group then H is isomorphic to $\mathbb{Z}$.

Hint: Use Stallings Theorem.

iv. Assume now that G is torsion free. Show that all G_i's are finite (and hence trivial) to conclude that $G \cong F_s$.

v. Deduce that if a f.g. torsion free group K has a finite index free subgroup then K is free.

Project 4. The objective of this project is to show that hyperbolic groups have finitely many conjugacy classes of finite subgroups.

Let X be a δ-hyperbolic geodesic metric space where $\delta > 0$. If A is a subset of bounded diameter of X we define the *radius* of A by

$$R = \inf\{r : A \subset B(x, r) \text{ for some } x\}.$$

i) Show that if $D = diam(A)$ and for some $x, y \in A$, $d(x, y) = D$ then $A \subset B(m, D/2 + \delta)$ where m is the midpoint of a geodesic joining x, y.

ii) Show that if $A \subset B(a, R)$ then $d(a, m) \leq 2\delta$. This shows that one can define the 'center' of a set up to 'finite error'.

iii) Give an example of a geodesic metric space X and a subset $A \subset X$ such that $diam(A) = radius(A)$.

iv) Let G be a δ-hyperbolic group and let A be a finite subgroup of G. Show that the cardinality of A is bounded by the cardinality of the ball of radius $2\delta + 1$ in G.

v) Show that the cardinality of conjugacy classes of finite subgroups of G is bounded by the number of subsets of $B(e, 2\delta + 1)$.

Hint: Note that if A, B are subgroups and $v \in G$ then $Av = Bv$ implies that $A = B$. Note also that if $AX = X$ then $gAg^{-1}(gX) = gX$ for any subset of G.