Geometric Group Theory

Problem Sheet 0-Solutions

1. Show that a subgroup of index 2 is normal.

Solution. If H is a subgroup of index 2 of G then for any $g \notin H$ we have $G = H \cup gH = H \cup Hg$ so $gH = Hg$ and H is normal.

2. Let A, B be finite index subgroups of G. Show that $A \cap B$ is a finite index subgroup of G.

Solution.

We show first that if $K < H < G$ then $|G : K| = |G : H| \cdot |H : K|$: Say $G = \bigcup Ha_i, H = \bigcup Kb_j$ then

$$G = \bigcup Kb_j a_i. \text{ (disjoint union)}$$

Indeed assume $Kb_j a_i = Kb_k a_l$ then $a_l a_i^{-1} \in H$ so $a_l = a_i$ and $b_k = b_j$.

We remark now that $|A : A \cap B|$ is finite since $|G : B|$ is finite and the map $a(A \cap B) \to aB$ is 1-1.

So $|G : A \cap B| = |G : A| \cdot |A : A \cap B| < \infty$.

3. Let G be a finitely generated group and let H be a subgroup of G of finite index. Show that H is finitely generated.

Solution. Let $A = \{a_1, ..., a_n\}$ be generators of G and let $X = \{x_1 = 1, ..., x_k\}$ be right coset representatives for H in G. Consider the set

$$S = \{x_i a_j x_k^{-1} : x_i, x_k \in X, a_j \in A, \text{ such that } x_i a_j x_k^{-1} \in H\}$$

If $g \in H$ then $g = g_1 ... g_r$, $(g_i \in A)$. Clearly there are $y_1, ..., y_{r-1} \in X$ such that all

$$g_1 y_1^{-1},\ y_1 g_2 y_2^{-1},\ \dots,\ y_{r-2} g_{r-1} y_{r-1}^{-1}$$

lie in S. Note that

$$(g_1 y_1^{-1}) \cdot (y_1 g_2 y_2^{-1}) \cdot \dots \cdot (y_{r-2} g_{r-1} y_{r-1}^{-1}) \cdot (y_{r-1} g_r) = g_1 ... g_r \in H$$

It follows that $y_{r-1} g_r \in S$, so S is a finite set of generators of H.

4. Show that if G is a finitely generated group such that every element of G has order 2 then G is finite.

Solution. Let $a, b \in G$. Then $abab = 1$ so $ab = ba$ and G is abelian. Since G is finitely generated $G \cong \mathbb{Z}_2^n$ for some $n \geq 0$.

5. Let H be a finite index subgroup of G. Show that there is a normal finite index subgroup of G, N such that $N \subset H$.

Solution. Consider the action of G on the left cosets aH where $g(aH) = (ga)H$. If $|G : H| = n$ this action induces a homomorphism from G to the finite symmetric group S_n so its kernel is a finite index normal subgroup N which is clearly contained in H. Since $|S_n| = n!$, $|G : N| \leq n!$.

6. Let G be a finitely generated group. Show that G has finitely many subgroups of index n. (*hint:* use the previous exercise).

Solution By the previous exercise any subgroup H of index n contains a normal subgroup N of index bounded by $n!$. So there is a homomorphism $f : G \to G/N$. where $|G/N| \leq n!$ and $\ker f \subseteq H$.

Consider the set of homomorphisms from G to groups of order at most $n!$. Since G is finitely generated and a homomorphism is given by assigning values to generators there are finitely many such homomorphisms say $f_1, ..., f_k$. If H_1, H_2 are subgroups of index n then for some f_i, $f_i(H_1) \neq f_i(H_2)$. Indeed take f_i such that $N = \ker f_i \subseteq H_1$. If $h_2 \in H_2 - H_1$ and $f_i(h_2) = f_i(h_1)$ with $h_1 \in H_1$ then $h_2 h_1^{-1} \in N$ so $h_2 \in N h_1 \subset H_1$, a contradiction.

In other words the map $H \to (f_1(H), ..., f_k(H))$ is 1-1.