Exercise sheet 1*

(for the class in W2)

Exercise 1.1 (Yoneda's lemma)**.** *Let $\mathcal{C}$ be a category. Let $\mathbf{Sets}^{\mathcal{C}^{\mathrm{opp}}}$ be the category of set-valued contravariant functors from $\mathcal{C}$ to $\mathbf{Sets}$. Prove that $\mathrm{Mor}_{\mathcal{C}}(\bullet, C)$ defines a contravariant functor $h_C : \mathcal{C} \to \mathbf{Sets}$ for each object C of $\mathcal{C}$. Prove that $h_\bullet$ defines a fully faithful functor $\mathcal{C} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{opp}}}$.*

Exercise 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Let $L : \mathcal{B} \to \mathcal{A}$ (resp. $R : \mathcal{A} \to \mathcal{B}$) be additive functors. Suppose that L is left adjoint to R (see Weibel, Introduction to Homological Algebra, Appendix A.6). Then L is right-exact and R is left-exact.*

Exercise 1.3. *Show that Theorem 1.4 implies Theorem 1.3. Let $\mathcal{A}$ be an abelian category with enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor to another abelian category. We say that an object A of $\mathcal{A}$ is F-**acyclic** if $R^k F(A) = 0$ for all $k > 0$. Show that if A is an object of $\mathcal{A}$ and $C^\bullet$ is a resolution of A, such that C^k is F-acyclic for all $k \in \mathbb{Z}$, then there is a natural isomorphism $R^k F(A) \simeq \mathcal{H}^k(F(C^\bullet))$ for all $k \in \mathbb{Z}$.*

Exercise 1.4 (optional)**.** *Show that an abelian group G is injective in the category **Ab** if G is divisible (ie for all $n \in \mathbb{Z}\backslash\{0\}$, the 'multiplication by n' map $G \to G$ is surjective). Show that the category **Ab** has enough injectives.*

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