Introduction to Schemes

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Prerequisites & Bibliography

- Chapters I, II and III (ie pp. 1–50) of [AM69] (basic definitions of commutative algebra).
- Appendix A (ie pp. 417–432) of [Wei94] (basic definitions of category theory).
- Section 1, 2 and 3 of Chapter I (ie pp. 1–15) of [Wei94] (basic definitions of homological algebra).
- The first section of Chapter 3, par. 5 (ie pp. 438–443) of [GH94] (the cohomological spectral sequence of a double complex).
- The definition of a topological space.

Bibliography

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

Further reading

[EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000.

[GW10] Ulrich Görtz and Torsten Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221 (French).

[GD] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique.* Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1960-1967).

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Lan02] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné; Oxford Science Publications.

[Mum99] David Mumford, *The red book of varieties and schemes*, Second, expanded edition, Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, Berlin, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians; With contributions by Enrico Arbarello.

Cohomology

Let $\mathcal{A}$ be an abelian category. An object I of $\mathcal{A}$ is called *injective* if the contravariant functor

$$\mathrm{Hom}_{\mathcal{A}}(\bullet, I) : \mathcal{A} \to \mathbf{Ab}$$

is exact.

Let $A^\bullet$ be a cochain complex in $\mathcal{A}$, which is bounded below. An *injective resolution* of $A^\bullet$ is a cochain complex in A

$$I^\bullet : \ldots \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \ldots$$

consisting of injective objects and such that:

- $I^\bullet$ is bounded below;
- there is a morphism of complex $A^\bullet \to I^\bullet$, which is a quasi-isomorphism.

If every cochain complex $A^\bullet$ in $\mathcal{A}$, which is bounded below, has an injective resolution, we say that $\mathcal{A}$ has *enough injectives.*

Let $(B^\bullet, d_B^\bullet)$ and $(C^\bullet, d_C^\bullet)$ be cochain complexes in $\mathcal{A}$.

Let $f^\bullet, g^\bullet : B^\bullet \to C^\bullet$ be two morphisms of complexes.

A *homotopy* $k^\bullet$ between $f^\bullet$ and $g^\bullet$ is a collection of morphisms

$$k^i : B^i \to C^{i-1}$$

$(i \in \mathbb{Z})$ such that

$$f^i - g^i = d_C^{i-1} \circ k^i + k^{i+1} \circ d^i$$

for all $i \in \mathbb{Z}$.

Lemma

The homotopy relation is an equivalence relation on complexes. If $f^\bullet$ and $g^\bullet$ as above are homotopic then $\mathcal{H}^k(f^\bullet) = \mathcal{H}^k(g^\bullet)$ for all $k \in \mathbb{Z}$, ie $f^\bullet$ and $g^\bullet$ induce the same morphisms in homology.

Lemma

Let $\phi : A \to B$ be a morphism of objects of $\mathcal{A}$. Let $I^\bullet$ (resp. $J^\bullet$) be an injective resolution of A (resp. B). Then there is a morphism of complexes $I^\bullet \to J^\bullet$, which is compatible with the morphisms $A \to I^0$, $B \to J^0$ and ϕ. Any two such morphisms are homotopic.

Let $\mathcal{B}$ be another abelian category.

Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant functor. We say that F is *additive* if for all objects A, B of $\mathcal{A}$, the map

$$\mathrm{Mor}(A, B) \to \mathrm{Mor}(F(A), F(B))$$

is a map of abelian groups.

We say that F is *left exact* if for any exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, the sequence

$$0 \to F(A') \to F(A) \to F(A'')$$

is also exact.

Suppose that $\mathcal{A}$ has enough injectives.

If $F : \mathcal{A} \to \mathcal{B}$ is a covariant left exact additive functor, we may for all $i \in \mathbb{Z}$ define a functor $R^i F$ by the following recipe.

For A and object in $\mathcal{A}$, let $I^\bullet$ be a injective resolution of A.

We define

$$R^i F(A) := \mathcal{H}^i(F(I^\bullet))$$

By the above lemmata, $\mathcal{H}^i(F(I^\bullet))$ is well-defined up to unique isomorphism and $R^i F : \mathcal{A} \to \mathcal{B}$ is an additive functor, called the i-th *right derived* functor of F.

Let $\mathcal{A}$ be an abelian category with enough injectives.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor to another abelian category.

Theorem

For any short exact sequence

$$0 \to A' \to A \to A'' \to 0 \tag{1}$$

there is a canonical 'long' exact sequence

$$0 \to R^0F(A') \to R^0F(A) \to R^0F(A'') \to R^1F(A') \to R^1F(A) \to \dots$$

which is naturally functorial in the short exact sequence (1).

Sheaves

Let X be a topological space.

Denote the category of abelian groups by $\mathbf{Ab}$.

Let $\mathrm{Top}(X)$ be the category whose objects are the open sets of X and whose arrows are the inclusion maps.

Definition

A presheaf F (of abelian groups) on X is a contravariant functor $F : \mathrm{Top}(X) \to \mathbf{Ab}$.

The presheaves on X naturally form a category, whose arrows are the natural transformations of functors.

If $U \to V$ is an inclusion of open subsets of X and $s \in F(V)$, we write

$$s|_U := F(U \to V)(s).$$

A *sheaf* on X is a presheaf F on X, with the following properties.

Let $(U_i \in \mathrm{Top}(X))$ be a family of open subsets of X. Then

- if $s \in F(\bigcup_i U_i)$ and $s|_{U_i} = 0$ for all indices i, then $s = 0$;
- if for all indices i we are given $s_i \in F(U_i)$ and

  $$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

  for all i, j then there is a unique element $s \in F(\bigcup_i U_i)$ such that $s|_{U_i} = s_i$ for all i.

[EL1]

Let F be a presheaf on a topological space X. Let $x \in X$.

Definition

The stalk of F at x is

$$F_x := \varinjlim_{U \in \mathrm{Top}(X),\ x \in U} F(U)$$

If $x \in X$ and $U \in \mathrm{Top}(X)$ contains x then for any $s \in F(U)$, we write s_x for the image of s in F_x.

If $\phi : F \to G$ is a morphism of presheaves on X, there is a unique map of abelian groups $\phi_x : F_x \to G_x$ such that for any $s \in F(U)$ and $U \in \mathrm{Top}(X)$ containing x, we have $\phi_x(s_x) = (\phi(s))_x$.

Let F be a presheaf on a topological space X. There is sheaf F^+ on X and a natural transformation

$$F \to F^+$$

uniquely defined by the following property: if G is a sheaf on X and $F \to G$ is a natural transformation, then there is a **unique** natural transformation $F^+ \to G$ such that the diagram

$$\begin{tikzcd} F^+ \arrow[r] & G \\ F \arrow[u] \arrow[ur] & \end{tikzcd}$$

commutes. The sheaf F^+ is called the *sheafification* of F.

If $\phi : F \to G$ is a morphism of sheaves on a topological space X, then we define the *kernel* $\ker(\phi)$ of ϕ as the presheaf

$$U \in \mathrm{Top}(X) \ \mapsto \ \ker(\phi(U))$$

This presheaf is a sheaf (exercise).

We define the *cokernel* $\mathrm{coker}(\phi)$ of ϕ as the sheafification of the presheaf

$$U \in \mathrm{Top}(X) \ \mapsto \ \mathrm{coker}(\phi(U))$$

Proposition-Definition

Let X be a topological space. The category $\mathbf{Ab}(X)$ of sheaves on X is an abelian category. If $\phi : F \to G$ is a morphism of sheaves, then the categorical kernel (resp. cokernel) of ϕ is canonically isomorphic to $\ker(\phi)$ *(resp.* $\operatorname{coker}(\phi)$*). A cochain complex*

$$\cdots \to F^{i-1} \to F^{i} \to F^{i+1} \to \dots$$

is exact in $\mathbf{Ab}(X)$ if and only for any $x \in X$, the corresponding sequence of stalks

$$\cdots \to F_x^{i-1} \to F_x^{i} \to F_x^{i+1} \to \dots$$

is exact.

Let $f : X \to Y$ be a continuous map of topological spaces.

For F a sheaf on X, we define the presheaf $f_*(F)$ by the formula

$$V \in \mathrm{Top}(Y) \;\mapsto\; F(f^{-1}(V))$$

The presheaf $f_*(F)$ is a sheaf (easy). This gives rise to an additive functor $\mathbf{Ab}(X) \to \mathbf{Ab}(Y)$.

For F a sheaf on Y, we define the sheaf $f^{-1}(F)$ as the sheafification of the presheaf on X given by the formula

$$U \in \mathrm{Top}(X) \;\mapsto\; \varinjlim_{V \in \mathrm{Top}(Y), V \supseteq f(U)} F(V)$$

Again, this leads to an additive functor $\mathbf{Ab}(Y) \to \mathbf{Ab}(X)$.

Proposition

The functor f^{-1} is left adjoint to the functor f_*.

The fact that f^{-1} and f_* are adjoint to each other formally implies that f_* is left exact and that f^{-1} is right exact. See Exercise.

If (F_i) is a family of sheaves on a topological space X, we define the presheaf $\prod_i F_i$ by the formula

$$U \in \mathrm{Top}(X) \ \mapsto \ \prod_i F_i(U)$$

where $\prod_i F_i(U)$ is the product of the abelian groups $F_i(U)$ (ie the cartesian product of the sets $F_i(U)$, endowed with the evident group structure). It can easily be verified that the presheaf $\prod_i F_i$ is a sheaf. By construction, if G is another sheaf on X, we have an identification

$$\mathrm{Mor}(G, \prod_i F_i) \simeq \prod_i \mathrm{Mor}(G, F_i)$$

Theorem

Let X be a topological space. The category $\mathbf{Ab}(X)$ *has enough injectives.*

Proof. We shall use the fact that **Ab** is a category with enough injectives.

Let F be a sheaf on X.

We shall construct an injective sheaf I and a monomorphism $F \to I$.

For each $x \in X$, choose an injective abelian group I_x and an injection $\iota_x : F_x \to I_x$. Denote also by x the inclusion map $x \to X$, where x is viewed as a topological space. Define

$$I := \prod_{x \in X} x_*(I_x)$$

Note that by construction we have for all $U \in \mathrm{Top}(X)$ an isomorphism

$$I(U) \simeq \prod_{x \in U} I_x$$

which is compatible with restrictions to smaller open sets.

In particular, we may define a morphism $F \to I$ by the formula

$$s \in F(U) \ \mapsto \ \prod_{x \in U} \iota_x(s_x)$$

This morphism is a monomorphism: if the image of $s \in F(U)$ vanishes, then $s_x = 0$ for all $x \in U$; hence $s = 0$ by the first sheaf property.

Now let

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of sheaves on X. We wish to show that the corresponding sequence

$$0 \to \mathrm{Mor}(F'', I) \to \mathrm{Mor}(F, I) \to \mathrm{Mor}(F', I) \to 0 \qquad (2)$$

is exact. Now we have natural isomorphisms

$$\begin{aligned} \mathrm{Mor}(F, I) &\simeq \mathrm{Mor}(F, \prod_{x\in X} x_*(I_x)) \simeq \prod_{x\in X} \mathrm{Mor}(F, x_*(I_x)) \\ &\simeq \prod_{x\in X} \mathrm{Mor}(x^{-1}(F), I_x) \simeq \prod_{x\in X} \mathrm{Mor}(F_x, I_x). \end{aligned}$$

Hence the sequence (2) is isomorphic to the product over all $x \in X$ of the sequences

$$0 \to \mathrm{Mor}(F''_x, I_x) \to \mathrm{Mor}(F_x, I_x) \to \mathrm{Mor}(F'_x, I_x) \to 0$$

which are exact because the I_x are injective abelian groups.QED

Cohomology of sheaves

The functor

$$\Gamma(X, \bullet) : \mathbf{Ab}(X) \to \mathbf{Ab}$$

described by the formula

$$\Gamma(X, F) := F(X)$$

is left exact.

More generally, let $f : X \to Y$ be a continuous map of topological spaces. The functor

$$f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$$

is left exact.

We shall often write $H^i(X, \bullet)$ for the i-th right derived functor $R^i\Gamma(X, \bullet)$ of $\Gamma(X, \bullet)$. [EL2]

Let $f : X \to Y$ be a continuous map of topological spaces.

Let F be a sheaf on X.

The following proposition can be proven by using the resolutions used when proving that $\mathbf{Ab}(X)$ has enough injectives for any topological space X.

Proposition

Let $V \in \mathrm{Top}(Y)$. Let $U := f^{-1}(V)$ and let $u : U \to X$, $v : V \to Y$ be the inclusion maps. Let $f_V : U \to V$ be the natural map. For all $i \geqslant 0$, we have canonical isomorphisms

$$v^{-1}(R^i f_*(F)) \simeq R^i f_{V,*}(u^{-1}(F)).$$

and these isomorphisms are natural in F.

Cech cohomology. Let F be a sheaf on a topological space X.

Let I be a finite set and let $(U_{i\in I})$ be a covering of X by open sets indexed by I.

We shall use the short-hand $i_0 \dots i_p$ for $(i_0, \dots, i_p) \in I^{\{0,\dots,p\}}$.

For $i_0, \dots i_p \in I$, we define

$$U_{i_0 \dots i_p} := U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_p}$$

and we let $j_{i_0 \dots i_p} : U_{i_0 \dots i_p} \to X$ be the inclusion map.

For all $p \geqslant 0$, let

$$\underline{C}^p((U_i), F) := \bigoplus_{i_0 \dots i_p} j_{i_0 \dots i_p, *}(j^{-1}_{i_0 \dots i_p}(F))$$

Define

$$d^p : \underline{C}^p((U_i), F) \to \underline{C}^{p+1}((U_i), F)$$

by

$$d^p(\bigoplus_{i_0 \dots i_p} \alpha_{i_0 \dots i_p}) = \sum_{k=0}^{p+1} (-1)^k \bigoplus_{i_0 \dots i_{p+1}} \alpha_{i_0 \dots \hat{i_k} \dots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}} \cap V}$$

where $V \in \mathrm{Top}(X)$ and

$$\alpha_{i_0 \dots i_p} \in j_{i_0 \dots i_p, *}(j^{-1}_{i_0 \dots i_p}(F))(V) = F(U_{i_0 \dots i_p} \cap V)$$

The hat symbol $\hat{\cdot}$ signifies that the term under the hat is omitted.

Furthermore, we define a morphism

$$d : F \to \underline{C}^0((U_i), F) = \bigoplus_i j_{i,*} j_i^{-1}(F)$$

by taking the direct sum of the natural morphisms

$$F \to j_{i,*} j_i^{-1}(F)$$

Theorem

The sequence of sheaves

$$0 \to F \xrightarrow{d} \underline{C}^0((U_i), F) \xrightarrow{d^0} \underline{C}^1((U_i), F) \xrightarrow{d^1} \dots$$

is an exact cochain complex.

Glueing sheaves.

Suppose given (U_i) an open covering of topological space X. If $j : U \to X$ is an open subset of X and F is a sheaf on X, we shall often write $F|_U$ instead of $j^{-1}(F)$.

Suppose given on U_i a sheaf F_i. Suppose given isomorphisms $\phi_{ij} : F_i|_{U_i \cap U_j} \xrightarrow{\sim} F_j|_{U_i \cap U_j}$ for all indices i, j, satisfying the properties (1), (2), (3) below.

(1) ϕ_{ii} is the identity;

(2) $\phi_{ji} = \phi_{ij}^{-1}$;

(3) $\phi_{ik}|_{U_i \cap U_j \cap U_k} = \phi_{jk} \circ \phi_{ij}|_{U_i \cap U_j \cap U_k}$.

for all indices i, j, k.

Proposition

Given sheaves F_i on U_i and isomorphisms

$$\phi_{ij} : F_i|_{U_i \cap U_j} \xrightarrow{\sim} F_j|_{U_i \cap U_j}$$

satisfying (1), (2), (3) above, there up to unique isomorphism a sheaf F on X with the following properties. There are isomorphisms

$$\psi_i : F|_{U_i} \xrightarrow{\sim} F_i$$

such that the natural isomorphism

$$(\psi_j^{-1}|_{U_i \cap U_j}) \circ \phi_{ij} \circ (\psi_i|_{U_i \cap U_j})$$

is the isomorphism

$$(F|_{U_i})|_{U_i \cap U_j} \simeq (F|_{U_j})|_{U_i \cap U_j}.$$

Flasque sheaves.

Let X be a topological space and let F be a sheaf on X.

The sheaf F is called *flasque* if for all $U, V \in \mathrm{Top}(X)$ such that $U \subseteq V$, the natural map $F(V) \to F(U)$ is surjective.

Lemma

If I is an injective sheaf on X, then I is flasque.

Proposition

If F is flasque then $H^k(X, F) = 0$ for all $k > 0$.

Ringed spaces

A *ringed space* is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. The ringed space $(X, \mathcal{O}_X)$ is said to be *locally ringed* if the stalks $\mathcal{O}_{X,x}$ are local rings for all $x \in X$. In that case we will often write $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ for the maximal ideal and $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ for the residue field of $\mathcal{O}_{X,x}$.

A *morphism of ringed spaces* $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a morphism of sheaves of rings $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed, we say that $(f, f^\#)$ is *local* or that it is *a morphism of locally ringed spaces* if for all $x \in X$, the induced map of stalks $\mathcal{O}_{f(x)} \to \mathcal{O}_x$ is a local morphism of rings.

Recall that a morphism of local rings $\phi : R \to T$ is said to be *local* if $\phi^{-1}(\mathfrak{m}_T) = \mathfrak{m}_R$. Here $\mathfrak{m}_T$ (resp. $\mathfrak{m}_R$) is maximal ideal of T (resp. R).

If $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^{\#}) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of ringed spaces, the composition

$$(h, h^{\#}) = (g, g^{\#}) \circ (f, f^{\#}) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$$

is defined in the following way. We let $h := g \circ f$ (as maps). The morphism of sheaves $h^{\#} : \mathcal{O}_Z \to h_*(\mathcal{O}_X)$ is defined as the unique morphism $h^{\#}$ making the following diagram commutative:

$$\begin{tikzcd}
g_*(\mathcal{O}_Y) \arrow[r, "g_*(f^{\#})"] & g_*(f_*(\mathcal{O}_X)) \\
\mathcal{O}_Z \arrow[u, "g^{\#}"] \arrow[r, "h^{\#}"] & (g \circ f)_*(\mathcal{O}_X) = h_*(\mathcal{O}_X) \arrow[u, "\simeq"']
\end{tikzcd}$$

[EL3]

Affine schemes

Let R be a ring. We define $\mathrm{Spec}(R)$ as the set of prime ideals of R. If $\mathfrak{a} \subseteq R$ is an ideal, we define

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, \mathfrak{p} \supseteq \mathfrak{a}\}$$

The symbol $V(\bullet)$ has the following properties:

- $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b})$;
- $\bigcap_{i \in I} V(\mathfrak{a}_i) = V(\sum_i \mathfrak{a}_i)$;
- $V(R) = \emptyset$; $V((0)) = \mathrm{Spec}(R)$.

As a consequence the sets $V(\mathfrak{a})$ ($\mathfrak{a}$ an ideal of R) form the closed sets of a topology on $\mathrm{Spec}(R)$. This topology is called the *Zariski topology*. The closed points in $\mathrm{Spec}(R)$ are precisely the maximal ideals of R.

Lemma

Let $f \in R$. The set

$$D_f(R) = D_f = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \notin \mathfrak{p}\}$$

is open in $\operatorname{Spec}(R)$. *The open sets of* $\operatorname{Spec}(R)$ *of the form* D_f *form a basis for the Zariski topology of* $\operatorname{Spec}(R)$. *The topology of* $\operatorname{Spec}(R)$ *is quasi-compact.*

The open sets of the form D_f are often called *basic open sets*.

Recall that a set B of open sets of a topological space X is said to be a *basis* for the topology of X if every open set of X can be written as a union of open sets in B.

We wish to make Spec(R) into a locally ringed space.

We define a sheaf of rings on Spec(R) as follows. For U open in Spec(R), let

$$\begin{aligned}&\mathcal{O}_{\mathrm{Spec}(R)}(U) := \\ &\{s : U \to \coprod_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{\mathfrak{p}} \mid \text{for all } \mathfrak{p} \in U \text{ we have } s(\mathfrak{p}) \in R_{\mathfrak{p}} \\ &\text{and for all } \mathfrak{p} \in U \text{ there is } a, r \in R \text{ and } V \in \mathrm{Top}(U) \\ &\text{such that } D_r(R) \supseteq V,\, \mathfrak{p} \in V \text{ and } s(\mathfrak{q}) = \frac{a}{r} \text{ for all } \mathfrak{q} \in V\}\end{aligned}$$

This formula clearly defines a sheaf on rings on Spec(R).

Proposition

- *For all* $r \in R$, *we have a canonical isomorphism*
$$\mathcal{O}_{\mathrm{Spec}(R)}(D_r(R)) \simeq R_r.$$
- *There is a natural isomorphism*
$$\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} \simeq R_{\mathfrak{p}}$$
for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

We have now associated with any ring R a locally ringed space

$$(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$$

and we have associated with any morphism $\phi : R \to T$ of rings a morphism of ringed spaces

$$(\mathrm{Spec}(\phi), \phi^{\#}),$$

which can easily be shown to be local using the previous Proposition.

We have in fact defined a *contravariant* functor from the category of rings to the category of locally ringed spaces.

Lemma

This functor is fully faithful.

Proof. We start with a morphism of locally ringed spaces

$$(f, f^{\#}) : (\mathrm{Spec}(T), \mathcal{O}_{\mathrm{Spec}(T)}) \to (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}).$$

We are thus given a morphism of sheaves of rings

$$\mathcal{O}_{\mathrm{Spec}(R)} \to f_*(\mathcal{O}_{\mathrm{Spec}(T)})$$

and thus a morphism of rings

$$\phi : R \simeq \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \to f_*(\mathcal{O}_{\mathrm{Spec}(T)})(\mathrm{Spec}(R)) \simeq T$$

we shall be done if we can show that $(f, f^{\#}) = (\mathrm{Spec}(\phi), \phi^{\#})$.

We shall first show that $f = \mathrm{Spec}(\phi)$.

We need to show that $\phi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(T)$.

Now we know that the morphism of rings

$$f^{\#}_{\mathfrak{p}} : \mathcal{O}_{\mathrm{Spec}(R), f(\mathfrak{p})} \to \mathcal{O}_{\mathrm{Spec}(T), \mathfrak{p}}$$

is local (because $(f, f^{\#})$ is a morphism of *locally* ringed spaces). This morphism fits in a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[d, "l_{f(\mathfrak{p})}"] & T \arrow[d, "l_{\mathfrak{p}}"] \\
\mathcal{O}_{\mathrm{Spec}(R), f(\mathfrak{p})} \arrow[r, "f^{\#}_{\mathfrak{p}}"] & \mathcal{O}_{\mathrm{Spec}(T), \mathfrak{p}}
\end{tikzcd}$$

We compute

$$\begin{aligned}\phi^{-1}(\mathfrak{p}) &= \phi^{-1}(l_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathcal{O}_{\mathrm{Spec}(T),\mathfrak{p}}})) = l_{f(\mathfrak{p}}^{-1}(f_{\mathfrak{p}}^{\#,-1}(\mathfrak{m}_{\mathcal{O}_{\mathrm{Spec}(T),\mathfrak{p}}})) \\ &= l_{f(\mathfrak{p})}^{-1}(\mathfrak{m}_{\mathcal{O}_{\mathrm{Spec}(R),f(\mathfrak{p})}}) = f(\mathfrak{p}).\end{aligned}$$

Here we have used the fact that $f_{\mathfrak{p}}^{\#}$ is local in the third equality.

The diagram also shows that $f_{\mathfrak{p}}^{\#} = \phi_{\mathfrak{p}}$.

Hence, we see that the morphisms of sheaves $\phi^{\#}$ and $f^{\#}$ coincide on the stalks.

This shows that there are equal. QED

[EL4]

A locally ringed space isomorphic to a space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called *an affine scheme.*

We shall write **Aff** for the category of affine schemes and **CRings** for the category of unital commutative rings.

A *scheme* is a locally ringed space X such that every point x in X has an open neighbourhood U, which is isomorphic to an affine scheme as a locally ringed space.

A *morphism of schemes* is a morphism of locally ringed spaces.

We shall write **Schemes** for the category of schemes.

A scheme X is called *locally noetherian* it is has an open covering (U_i) such that each U_i is isomorphic to an affine scheme

$$(\mathrm{Spec}(R_i), \mathcal{O}_{\mathrm{Spec}(R_i)}),$$

where R_i is a noetherian ring.

Recall that a ring is *noetherian*, if every ideal of R is finitely generated as an R-module.

Proposition

A scheme X is locally noetherian if and only if for any open subset U of X, which is isomorphic to an affine scheme $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ as a locally ringed space, the ring R is noetherian.

A scheme X is called *noetherian* if it is quasi-compact as a topological space and locally noetherian.

A scheme X is *reduced* if for all $U \in \mathrm{Top}(X)$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

A scheme X is *integral*, if for all $U \in \mathrm{Top}(X)$, the ring $\mathcal{O}_X(U)$ is a domain (also called an integral ring).

An *open affine covering* $(U_{i\in I})$ of X is a family of open subsets U_i of X such that

- $\bigcup_i U_i = X$;
- if U_i is endowed with the structure of locally ringed space coming from X, then U_i is an affine scheme.

Properties of morphisms of schemes.

Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes.

- $(f, f^{\#})$ is *quasi-compact* if there is an open affine covering (V_i) of Y such that $f^{-1}(V_i)$ is quasi-compact for all i.
- $(f, f^{\#})$ is *locally of finite type* if f there is a an open affine covering (V_i) of Y and for each i an open affine covering (U_{ij}) of $f^{-1}(V_i)$ such that $\mathcal{O}_X(U_{ij})$ is a finitely generated $\mathcal{O}_Y(V_i)$-algebra via the morphism $(f, f^{\#})$.
- $(f, f^{\#})$ is *of finite type* of it is quasi-compact and locally of finite type.

- $(f, f^{\#})$ is *a closed immersion* if the image of f is closed, f is a homeomorphism of X onto $f(X)$ and the morphism of sheaves

  $$f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$$

  is surjective.

  We then say that X is a *closed subscheme* of Y via $(f, f^{\#})$ or simply that $f(X)$ is a closed subscheme of Y.

- $(f, f^{\#})$ is *an open immersion* if $f(X)$ is open, f is a homeomorphism onto its image and the mapping of stalks

  $$f^{\#}_y : \mathcal{O}_y \to (f_* \mathcal{O}_X)_y$$

  is an isomorphism for all $y \in f(X)$.

  We then say that X is an *open subscheme* of Y via $(f, f^{\#})$ or simply that $f(X)$ is an open subscheme of X.

Glueing schemes. Suppose given (U_i) a family of schemes and for each pair of indices ij an open subscheme $U_{ij} \to U_i$

Suppose given isomorphisms $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ for all indices i, j, satisfying the properties (1), (2), (3) below.

(1) $U_{ii} = U_i$;

(2) $\phi_{ij}(U_{ij} \cap U_{ik}) \subseteq U_{jk}$;

(3) $\phi_{ik}|_{U_{ij} \cap U_{ik}} = \phi_{jk} \circ \phi_{ij}|_{U_{ij} \cap U_{ik}}$ as morphisms $U_{ij} \cap U_{ik} \to U_k$.

for all indices i, j, k.

Proposition

There is up to unique isomorphism a scheme X with the following properties. There are open immersions $\psi_i : U_i \to X$ such that $\bigcup_i \psi_i(U_i) = X$ and such that $\psi_j \circ \phi_{ij} = \psi_i|_{U_{ij}}$.

Products.

Let $\mathcal{C}$ be a category. Let (C_i) $(i = 1, \ldots, n)$ be a finite family of objects in $\mathcal{C}$.

Recall that the *product*

$$C_1 \times \cdots \times C_n = \prod_i C_i$$

of the C_i (it it exists) is an object P of $\mathcal{C}$ together with arrows

$$\pi_i : P \to C_i$$

characterised by the following property. If P' is another object together with arrows $\pi'_i : P' \to C_i$ then there is a unique arrow $u : P' \to P$ such that $\pi_i \circ u = \pi'_i$ for all i.

If C is an object of $\mathcal{C}$, we shall write $\mathcal{C}/C$ for the following category.

The objects of $\mathcal{C}/C$ are morphisms $D \to C$ in $\mathcal{C}$.

A morphism from $\phi : D \to C$ to $\lambda : E \to C$ is a morphism $\mu : D \to E$ such that $\lambda \circ \mu = \phi$.

The morphism μ, viewed as a morphism in $\mathcal{C}$, is often called a *C-morphism.*

The category $\mathcal{C}/C$ is called the *category of C-objects* (associated with $\mathcal{C}$ and C).

One often writes $D \times_C E$ for the product of $D \to C$ and $E \to C$ in $\mathcal{C}/C$ (if it exists). It is sometimes called the *fibre product* of D and E over C.

Proposition

Let S be a scheme. Finite products exist in **Schemes**$/S$*.*

Notice that if R is a ring and A and B are two R-algebras, then the tensor product

$$A \otimes_R B$$

is the coproduct of A and B in the category of R-algebras. Hence

$$(\mathrm{Spec}(A \otimes_R B), \mathcal{O}_{\mathrm{Spec}(A \otimes B)})$$

is the product of $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ and $(\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)})$ in the category $\mathbf{Aff}/\mathrm{Spec}(R)$. The proof glues such objects together into a scheme.

[EL5]

Let X be a ringed space.

An *$\mathcal{O}_X$-module* or *sheaf in $\mathcal{O}_X$-modules* is an abelian sheaf F, together with a $\mathcal{O}_X(U)$-module structure on $F(U)$ for every open set $U \subseteq X$, subject to obvious compatibility properties with respect to inclusions $U \to V$ of open sets in X.

A *morphism of $\mathcal{O}_X$-modules* $F \to G$ is a morphism of abelian sheaves compatible with the $\mathcal{O}_X$-module structure in an obvious sense.

The $\mathcal{O}_X$-modules form an additive category $\mathrm{Mod}_{\mathcal{O}_X}(X)$, which is abelian.

Let F and G be $\mathcal{O}_X$-modules on X.

The tensor product $F \otimes_{\mathcal{O}_X} G$ is the sheaf generated by the presheaf on X given by the formula

$$U \mapsto F(U) \otimes_{\mathcal{O}_X(U)} G(U)$$

This sheaf has a unique structure of $\mathcal{O}_X$-module, such that the map

$$F(U) \otimes_{\mathcal{O}_X(U)} G(U) \to (F \otimes_{\mathcal{O}_X} G)(U)$$

is a map of $\mathcal{O}_X(U)$-modules for every $U \in \mathrm{Top}(X)$.

Suppose $f : X' \to X''$ is a continuous map of topological spaces and that X'' is ringed by the sheaf of rings $\mathcal{O}_{X''}$.

Let F be a sheaf in $\mathcal{O}_{X''}$-modules on X''.

The abelian sheaf $f^{-1}(\mathcal{O}_{X''})$ is then endowed with a canonical structure of sheaf of rings, as can be seen by looking at its definition.

Furthermore, the abelian sheaf $f^{-1}(F)$ inherits an obvious $f^{-1}(\mathcal{O}_{X''})$-module structure from the $\mathcal{O}_{X''}$-module structure of F on X''.

Let $f : Z \to X$ be a morphism of ringed spaces.

Let F be a $\mathcal{O}_X$-module. We define

$$f^*(F) := f^{-1}(F) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Z.$$

Here $\mathcal{O}_Z$ is viewed as a $f^{-1}(\mathcal{O}_X)$-module through the canonical map of sheaves of rings $f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Z$.

For each $U \in \mathrm{Top}(Z)$, the group

$$f^{-1}(F)(U) \otimes_{f^{-1}(\mathcal{O}_X)(U)} \mathcal{O}_Z(U)$$

has a $\mathcal{O}_Z(U)$-module structure, which comes from the action of $\mathcal{O}_Z(U)$ on the second factor.

There is a unique structure of $\mathcal{O}_Z$-module on $f^*(F)$ such that for all $U \in \mathrm{Top}(Z)$, the map

$$f^{-1}(F)(U) \otimes_{f^{-1}(\mathcal{O}_X)(U)} \mathcal{O}_Z(U) \to (f^{-1}(F) \otimes_{f^{-1}(\mathcal{O}_X)(U)} \mathcal{O}_Z)(U) = f^*(U)$$

is a map of $\mathcal{O}_Z(U)$-modules.

Let now F be a $\mathcal{O}_Z$-module.

The abelian sheaf $f_*(F)$ is naturally a sheaf in $f_*(\mathcal{O}_Z)$-modules.

Via the morphism of sheaves of rings $\mathcal{O}_X \to f_*(\mathcal{O}_Z)$, we may thus view $f_*(F)$ as a $\mathcal{O}_X$-module.

Lemma

The functor $f^* : \mathrm{Mod}_{\mathcal{O}_X}(X) \to \mathrm{Mod}_{\mathcal{O}_Z}(Z)$ *is left-adjoint to the functor*

$$f_* : \mathrm{Mod}_{\mathcal{O}_Z}(Z) \to \mathrm{Mod}_{\mathcal{O}_X}(X).$$

See Exercises.

Quasi-coherent sheaves.

Let R be a ring and let M be an R-module. We define a sheaf $\widetilde{M}$ on $\operatorname{Spec}(R)$ by the recipe

$$\widetilde{M}(U) := \{s : U \to \coprod_{\mathfrak{p} \in \operatorname{Spec}(R)} M_{\mathfrak{p}} \mid \text{for all } \mathfrak{p} \in U \text{ we have } s(\mathfrak{p}) \in M_{\mathfrak{p}}$$
$$\text{and for all } \mathfrak{p} \in U \text{ there is } a \in M,\ r \in R \text{ and } V \in \operatorname{Top}(U)$$
$$\text{such that } D_r(R) \supseteq V \supseteq \{\mathfrak{p}\} \text{ and } s(\mathfrak{q}) = \frac{a}{r} \text{ for all } \mathfrak{q} \in V\}$$

Notice that $\mathcal{O}_{\operatorname{Spec}(R)} = \widetilde{R}$.

The sheaf $\widetilde{M}$ carries an obvious $\mathcal{O}_{\mathrm{Spec}(R)}$-module structure.

Also, if $M \to N$ is a morphism of R-modules, there is an obvious associated morphism of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\widetilde{M} \to \widetilde{N}$.

We have thus defined a functor from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

(a) For all $r \in R$, we have a canonical isomorphism

$$\widetilde{M}(D_r(R)) \simeq M_r.$$

(b) If $t \in R$ and $t \in (r)$ then there is a commutative diagram

$$\begin{tikzcd} \widetilde{M}(D_r(R)) \arrow[r] \arrow[d, "\simeq"] & \widetilde{M}(D_t(R)) \arrow[d, "\simeq"] \\ M_r \arrow[r] & M_t \end{tikzcd}$$

where the vertical isomorphisms come from (a).

(c) There is a natural isomorphism $\widetilde{M}_{\mathfrak{p}} \simeq M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. This isomorphism fits in a commutative diagram

$$\begin{tikzcd} \widetilde{M}_{\mathfrak{p}} \arrow[r, "\simeq"] & M_{\mathfrak{p}} \\ \widetilde{M}(\operatorname{Spec}(R)) \arrow[u] \arrow[r, "\simeq"] & M \arrow[u] \end{tikzcd}$$

Here the vertical morphisms are the natural ones and the lower horizontal one comes from (b).

Corollary

The functor $\widetilde{\bullet}$ from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules is fully faithful and exact.

Let now X be a scheme.

Definition

Let F be a sheaf on $\mathcal{O}_X$-modules. The sheaf F is said to be quasi-coherent (resp. coherent) if there is an open affine covering (U_i) of X, such that $F|_{U_i} \simeq \widetilde{F(U_i)}$ (resp. $F|_{U_i} \simeq \widetilde{F(U_i)}$ and $F(U_i)$ is a finitely generated $\mathcal{O}_X(U_i)$-module).

The full subcategory of $\mathrm{Mod}(X)$, which are quasi-coherent, will be denoted $\mathfrak{Q}coh(X)$.

Lemma

Let $\phi : R \to T$ be a morphism of rings. Let M be a T-module. Then there is a natural isomorphism of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules

$$\mathrm{Spec}(\phi)_*(\widetilde{M}) \simeq \widetilde{M_0},$$

where M_0 is M viewed as an R-module via ϕ.

Proof. Notice that for all $r \in R$, there a natural isomorphisms of R_r-modules

$$\begin{aligned} & \mathrm{Spec}(\phi)_*(\widetilde{M})(D_r(R)) = \widetilde{M}(\mathrm{Spec}(\phi)^{-1}(D_r(R)) \\ = \quad & \widetilde{M}(D_{\phi(r)}(T)) \simeq M_{\phi(r)} \simeq M_{0,r} \end{aligned}$$

which are compatible with restrictions $D_r(R) \supseteq D_{r'}(R)$ for $r' \in (r)$. Now the lemma follows from the fact that the sets $D_r(R)$ form a basis for the topology of $\mathrm{Spec}(R)$ and the fact that $\mathrm{Spec}(\phi)_*(\widetilde{M})$ and $\widetilde{M_0}$ are both sheaves. QED

Proposition

The definition of a quasi-coherent (resp. coherent) sheaf is independent of the open affine covering appearing in its definition.

Proof. After simple logical reductions, we are reduced to the following problem.

Suppose $X = \mathrm{Spec}(R)$ is an affine scheme and let F be an $\mathcal{O}_X$-module on X. Let $(V_j = \mathrm{Spec}(R_{f_j}))$ be a covering of X by basic open sets. Suppose $F|_{V_j} \simeq \widetilde{M_j}$, where M_j is an R_{f_j}-module. Then $F \simeq \widetilde{M}$ for some R-module M.

Notice that we may suppose that the family (V_j) is finite, since X is quasi-compact. Notice also that $V_{j_1} \cap V_{j_2} = \mathrm{Spec}(R_{f_{j_1} f_{j_2}})$. In particular $V_{j_1} \cap V_{j_2}$ is also affine. Now look at the two first terms of the Cech complex associated with (V_j). These terms are in the essential image of the functor $\widetilde{\bullet}$ by the preceding lemma. Since the functor $\widetilde{\bullet}$ is exact, we are done. QED

Lemma (Deligne)

Let R be a noetherian ring and let M be an R-module. Let $\mathfrak{a}$ be an ideal of R. There is an isomorphism

$$\varinjlim_n \mathrm{Hom}_R(\mathfrak{a}^n, M) \simeq \widetilde{M}(\mathrm{Spec}(R)\backslash V(\mathfrak{a})),$$

which is natural in M.

The morphism arises from the isomorphism

$$\mathfrak{a}^n|_{\mathrm{Spec}(R)\backslash V(\mathfrak{a})} \simeq \mathcal{O}_{\mathrm{Spec}(R)\backslash V(\mathfrak{a})}.$$

Corollary

Let I be an injective module over R. Then $\widetilde{I}$ is a flasque sheaf.

Sheaves of ideals. Let X be a scheme. A subsheaf of $\mathcal{O}_X$ is called *a sheaf of ideals* on X.

Lemma

Let J be a quasi-coherent sheaf of ideals on X. There exists a closed immersion $(z, z^{\#}) : Z \to X$ such that $J = \ker(z^{\#})$. This immersion is unique up to unique isomorphism over X.

The proof is by glueing. See Exercises.

Lemma

Let $C_0 \subseteq X$ be a closed subset. Then there is a unique quasi-coherent ideal I_{C_0} in X, such that the image of the closed immersion $C \to X$ associated with I_{C_0} is C_0 and such that C is reduced.

The proof follows from the fact that the formation of the nil radical commutes with localisation. See exercises.

Permanence properties of quasi-coherent sheaves.

Let X be a ringed space and let (F_i) be a family of $\mathcal{O}_X$-modules on X. We write $\bigoplus_i F_i$ for the sheaf generated by the presheaf in $\mathcal{O}_X$-modules on X sending $U \in \mathrm{Top}(X)$ to $\bigoplus_i F_i(U)$.

Lemma

Let I be an index set. For any object (F_i) of $\mathrm{Mod}(X)^I$ and any object G in $\mathrm{Mod}(X)$, there is a canonical isomorphism

$$\mathrm{Mor}(\bigoplus_i F_i, G) \simeq \prod_i \mathrm{Mor}(F_i, G)$$

which is natural in (F_i) and G.

In categorical terms, Lemma 11 says that the direct sum is a categorical coproduct in the category $\mathrm{Mod}(X)$.

Lemma

Let X be a scheme and let (F_i) be a family of quasi-coherent sheaves on X. Then $\bigoplus_i F_i$ is quasi-coherent.

Proof. Let R be a ring and (M_i) be a family of R-modules. If $r \in R$, there is a functorial isomorphism $(\bigoplus_i M_i)_r \simeq \bigoplus_i M_{i,r}$. The Lemma follows from this. QED

A formal consequence of the last two lemmata is the following fact. Let R be a ring and let (M_i) be a family of R-modules. Then there is a functorial isomorphism of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules

$$\widetilde{(\bigoplus_i M_i)} \simeq \bigoplus_i \widetilde{M_i}$$

Proposition

Let $\phi : R \to T$ be a morphism of rings and let M be an R-module. Then $\mathrm{Spec}(\phi)^*(\widetilde{M})$ is a quasi-coherent sheaf.

Proof. First notice the following fact.

Let $(X, \mathcal{O}_X) := (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. Let G be a $\mathcal{O}_X$-module.

Then G is quasi-coherent if and only if there exist index sets I and J and exact sequence of $\mathcal{O}_X$-modules

$$\bigoplus_{i \in I} \mathcal{O}_X \to \bigoplus_{j \in J} \mathcal{O}_X \to G \to 0 \tag{3}$$

Indeed if G has a presentation (3) then by the above, we conclude that G is quasi-coherent.

On the other hand, if $G = \widetilde{M}$ for some R-module then we may choose a surjection $u : \bigoplus_{j \in J} R \to M$ and a surjection $\bigoplus_{i \in I} R \to \ker(u)$.

Applying the functor $\widetilde{\bullet}$, we then obtain a presentation (3).

Let $(Y, \mathcal{O}_Y) := (\mathrm{Spec}(T), \mathcal{O}_{\mathrm{Spec}(T)})$.

In view of the above fact and the fact that $\mathrm{Spec}(\phi)^*$ is right exact, we see that we are reduced to prove that there is an isomorphism

$$f^*(\bigoplus_i \mathcal{O}_X) \simeq \bigoplus_i \mathcal{O}_Y \tag{4}$$

To show this, first notice that there is an isomorphism $f^*(\mathcal{O}_X) \simeq \mathcal{O}_Y$. For this notice that we have canonical isomorphisms for any $\mathcal{O}_Y$-module G

$$\begin{aligned} \mathrm{Mor}_{\mathrm{Mod}(Y)}(f^*(\mathcal{O}_X), G) &\simeq \mathrm{Mor}_{\mathrm{Mod}(X)}(\mathcal{O}_X, f_*(G)) \\ &\simeq f_*(G)(X) \simeq G(Y) \simeq \mathrm{Mor}_{\mathrm{Mod}(Y)}(\mathcal{O}_Y, G) \end{aligned}$$

and thus $f^*(\mathcal{O}_X)$ and $\mathcal{O}_Y$ represent the same covariant functor. We conclude by appealing to Yoneda's lemma.

To prove that there is an isomorphism (4), we notice that there are functorial isomorphisms

$$\begin{aligned} &\mathrm{Mor}_{\mathrm{Mod}(Y)}(f^*(\bigoplus_i \mathcal{O}_X), G) \simeq \mathrm{Mor}_{\mathrm{Mod}(X)}(\bigoplus_i \mathcal{O}_X, f_*(G)) \\ \simeq \quad &\prod_i \mathrm{Mor}_{\mathrm{Mod}(X)}(\mathcal{O}_X, f_*(G)) = \prod_i f_*(G)(X) = \prod_i G(Y) \end{aligned}$$

Thus, we have functorial isomorphisms

$$\mathrm{Mor}_{\mathrm{Mod}(Y)}(\bigoplus_i f^*(\mathcal{O}_X), G) \simeq \mathrm{Mor}_{\mathrm{Mod}(Y)}(\bigoplus_i \mathcal{O}_Y, G) = \prod_i G(Y)$$

and thus again $\bigoplus_i \mathcal{O}_Y$ and $f^*(\bigoplus_i \mathcal{O}_X)$ represent the same covariant functor and must thus be isomorphic. QED

Corollary

There is a functorial isomorphism

$$\mathrm{Spec}(\phi)^*(\widetilde{M}) \simeq \widetilde{M \otimes_R T}.$$

Proof. Follows from the uniqueness of adjoint functors and from the fact that there is a functorial isomorphism

$$\mathrm{Mor}_R(N, M) \simeq \mathrm{Mor}_T(N \otimes_R T, M)$$

for any R-module N and T-module M. QED

Corollary

Let $f : X \to Y$ be a morphism of schemes. Let F be quasi-coherent sheaf on Y. Then $f^*(F)$ is also quasi-coherent.

[EL7]

Proposition

Let $f : X \to Y$ be a morphism of schemes. Suppose that X is noetherian. Let F be a quasi-coherent $\mathcal{O}_X$-module. Then $f_*(F)$ is also quasi-coherent.

Proof. We may assume wrog that Y is affine.

Let (U_i) be a finite open affine cover of X and for all i, j let U_{ijk} be a finite open affine cover of $U_i \cap U_j$ indexed by k. Looking at the beginning of the Cech complex and using the fact that f_* is left exact as a functor from $\mathrm{Mod}_{\mathcal{O}_X}(X)$ to $\mathrm{Mod}_{\mathcal{O}_Y}(Y)$, we see that there is an exact sequence

$$0 \to f_*(F) \to \bigoplus_i f_*(F|_{U_i}) \to \bigoplus_{i,j,k} f_*(F|_{U_{ijk}})$$

Thus we see that is sufficient to prove the proposition under the assumption that X is also affine, where it was already proven. QED

Proposition

Let $f : X \to Y$ be a morphism of schemes. Suppose that X is noetherian. Let F be a quasi-coherent sheaf on X.

Then the $\mathcal{O}_Y$-module $R^k f_*(F)$ is also quasi-coherent.

Proof. (sketch) F has a resolution by quasi-coherent flasque sheaves and thus the proposition follows from the previous proposition. QED

The cohomology of affine schemes.

Proposition

Let X be a noetherian affine scheme and let F be a quasi-coherent sheaf on X.

Then $H^k(X, F) = 0$ for all $k > 0$.

Proof. Suppose $X = \mathrm{Spec}(R)$. If I is an injective R-module, then $\widetilde{I}$ is flasque. Thus the proposition follows from the fact that $\Gamma(X, \bullet)$ is an exact functor from $\mathfrak{Q}coh(X)$ to the category of R-modules. QED

The following theorem is a converse.

Theorem (Serre)

Let X be a noetherian scheme and suppose that for all coherent sheaves F on X, we have $H^1(X, F) = 0$.

Then X is an affine scheme.

The proof of Serre's theorem uses the following lemmata.

Lemma

Let X be a noetherian scheme and let $f \in \Gamma(X, \mathcal{O}_X)$. Then there is a natural isomorphism $\Gamma(X, \mathcal{O}_X)_f \xrightarrow{\sim} \Gamma(X_f, \mathcal{O}_{X_f})$.

Corollary

Let X be a noetherian scheme and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ be such that $(f_1, \ldots, f_n) = \Gamma(X, \mathcal{O}_X)$.

If the open subschemes X_{f_i} are all affine, then X is affine.

Proof. (of the corollary). The canonical morphism

$$X \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$$

is an isomorphism. QED

Proof of Serre's theorem.

Let P be a closed point in X.

This exists because X is quasi-compact.

Let U be an open affine neighbourhood of P and let Y be the complement of U in X.

We view P, Y and $P \cup Y$ as reduced closed subschemes of X. Let I_P, I_Y and $I_{P \cup Y}$ be the corresponding quasi-coherent sheaves of ideals.

Note that we have canonically $\mathcal{O}_P(P) \simeq \kappa(P)$ and that this isomorphism describes the sheaf $\mathcal{O}_P$ entirely.

By construction, we have an exact sequence

$$0 \to I_{Y \cup P} \to I_Y \to \kappa(P) \to 0$$

where $\kappa(P)$ denotes the direct image of $\mathcal{O}_P$ by the closed immersion $P \to X$. The long cohomology sequence gives

$$\Gamma(X, I_Y) \to \Gamma(X, \kappa(P)) \to H^1(X, I_{Y \cup P})$$

and since by assumption $H^1(X, I_{Y \cup P}) = 0$, we get a surjection

$$\Gamma(X, I_Y) \to \Gamma(X, \kappa(P)).$$

Let $f \in \Gamma(X, I_Y)$ be such that the image of f in $\Gamma(X, \kappa(P)) \simeq \kappa(P)$ is 1. We view f as an element of $\Gamma(X, \mathcal{O}_X)$ via the natural inclusion

$$\Gamma(X, I_Y) \to \Gamma(X, \mathcal{O}_X).$$

By construction, we have that $P \in X_f$ and also that $X_f \subseteq U$.

In particular, X_f is affine, because it corresponds to a basic open set in U.

If $X \neq X_f$, we now repeat this reasoning for a closed point P_2 in $X \backslash X_f$ and we obtain $f_2 \in \Gamma(X, \mathcal{O}_X)$ such that $P_2 \in X_{f_2}$ and X_{f_2} is affine and we repeat it for $P_3 \in X \backslash X_f \cup X_{f_2}$ etc.

The sequence of the X_{f_i} must stop after a finite number of steps, and thus cover X, because X is a noetherian topological space.

We can thus exhibit a finite sequence $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that X_{f_i} is affine for all i and such that the X_{f_i} cover X.

By the Corollary above, we shall be able to conclude if we can show that the f_i generate $\Gamma(X, \mathcal{O}_X)$. To see this, consider the morphism of sheaves

$$\bigoplus_{i=1}^{n} \mathcal{O}_X \to \mathcal{O}_X$$

sending local sections $(s_1, \ldots, s_n)$ to $\sum_i f_i \cdot s_i$.

This morphism is surjective, because the X_{f_i} cover X. Using the assumptions we obtain a surjection

$$\bigoplus_{i=1}^{n} \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X).$$

In other words, the f_i generate $\Gamma(X, \mathcal{O}_X)$. QED

[EL8]

Affine spaces

Let $r \geqslant 0$. Consider the functor $\underline{\mathbb{A}}^r$ from **Schemes** to **Sets**, which associates with a scheme S the set of morphisms of sheaves

$$\phi : \bigoplus_{k=1}^{r} \mathcal{O}_S \to \mathcal{O}_S$$

Lemma

Let X be a scheme. The restriction of the functor $\underline{\mathbb{A}}^r$ to $\mathrm{Top}(X)$ is a sheaf of sets.

See Exercises.

Lemma

Let X, S be schemes. Let $h_S : \textbf{Schemes} \to \textbf{Sets}$ be the functor $\mathrm{Mor}(\bullet, S)$. Then the restriction of h_S to $\mathrm{Top}(X)$ is a sheaf of sets.

Proof: Glue! QED

Proposition-Definition

$\underline{\mathbb{A}}^r$ is representable by the scheme

$$\mathbb{A}^r := \mathrm{Spec}(\mathbb{Z}[X_1, \dots, X_r])$$

called the affine space of relative dimension r.

Proof. In view of the two last Lemmata, it is sufficient to construct an isomorphism between the restriction of the functor $h_{\mathbb{A}^r}$ to $\textbf{Aff}$ and the restriction of the functor $\underline{\mathbb{A}}^r$ to $\textbf{Aff}$.

• The restriction $h_{\mathbb{A}^r}|_{\mathbf{Aff}}$ of $h_{\mathbb{A}^r}$ to **Aff** in the language of rings is the functor

$$R \mapsto \mathrm{Mor}_{\mathbf{CRings}}(\mathbb{Z}[X_1, \ldots, X_r], R)$$

• The restriction $\underline{\mathbb{A}}^r|_{\mathbf{Aff}}$ of the functor $\underline{\mathbb{A}}^r$ to **Aff** in the language of rings is the functor

$$R \mapsto \mathrm{Mor}_{\mathbf{Sets}}(\{1, \ldots, r\}, R)$$

Now there is a natural transformation between $h_{\mathbb{A}^r}|_{\mathbf{Aff}}$ and $\underline{\mathbb{A}}^r|_{\mathbf{Aff}}$, which for every ring R maps $\mathrm{Mor}_{\mathbf{CRings}}(\mathbb{Z}[X_1, \ldots, X_r], R)$ to $\mathrm{Mor}_{\mathbf{Sets}}(\{1, \ldots, r\}, R)$, by sending

$$\phi \in \mathrm{Mor}_{\mathbf{CRings}}(\mathbb{Z}[X_1, \ldots, X_r], R)$$

to

$$\phi(X_\bullet).$$

This map is an isomorphism by the definition of polynomials. QED

Let R be a ring and let $X \to \operatorname{Spec}(R)$ be a scheme over R.

From the definitions, we see that to say that X is *locally of finite type over* R is the same as to say that there exists

- an open covering (U_i) of X by affine open subschemes;
- for each i, an $r(i) \in \mathbb{N}$ and a commutative diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[dr] & \mathbb{A}_R^{r(i)} := \mathbb{A}^{r(i)} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(R) \arrow[d] \\ & \operatorname{Spec}(R) \end{tikzcd}$$

where the vertical morphism is the natural one and the horizontal morphism is a *closed immersion*.

These closed immersions are in general not related to each other and one may wonder what kind of compatibilities could be required.

Projective spaces propose an answer to this question.

Projective spaces.

Let $r \geqslant 0$. Consider the functor $\underline{\mathbb{P}}^r$: **Schemes** $\to$ **Sets**, such that

$$\underline{\mathbb{P}}(S) := \{\text{iso. classes of surjective morphisms } \phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}\}$$

where $\mathcal{L}$ is locally free of rank 1. Here a surjective morphism

$$\phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}$$

is said to be isomorphic to a surjective morphism

$$\psi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{M}$$

if there is an isomorphism $\iota : \mathcal{L} \simeq \mathcal{M}$ such that $\iota \circ \phi = \psi$.

A sheaf, which is locally free of rank one, is often called *a line bundle.*

Theorem

The functor $\underline{\mathbb{P}}^r$ *is representable by a scheme* $\mathbb{P}^r$*, which is integral and of finite type over* $\mathrm{Spec}(\mathbb{Z})$*.*

In particular $\mathbb{P}^r$ is noetherian.

The scheme $\mathbb{P}^r$ is called *projective space of relative dimension* r.

Proof. Let K be the fraction field of the ring $\mathbb{Z}[X_0, \ldots, X_r]$. Let $i, j, k \in \{0, \ldots, r\}$. Define

$$R_i := \mathbb{Z}[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}] \subseteq K$$

and

$$R_{ij} := R_{i, \frac{X_j}{X_i}} \subseteq K.$$

Notice also that we have morphisms of R_i-algebras

$$R_{ij} \otimes_{R_i} R_{ik} \simeq R_{i, \frac{X_j}{X_i} \cdot \frac{X_k}{X_i}} \overset{\subseteq}{\to} K.$$

Furthermore, it is easy to verify that we have the following set-theoretic relations between subsets of K:

$$R_i = R_{ii},\ R_{ij} = R_{ji},\ R_i \subseteq R_{ij},\ R_{jk} \subseteq R_{i,\frac{X_j}{X_i}\cdot\frac{X_k}{X_i}}$$

In view of these identities and the fact that any diagram of inclusions of subrings of K commutes, we see that the schemes

$$U_i = \mathrm{Spec}(R_i)$$

and

$$U_{ij} = \mathrm{Spec}(R_{ij})$$

together with the open immersions $U_{ij} \to U_i$ and the isomorphisms $U_{ij} \simeq U_{ji}$ coming from the corresponding inclusions of rings, define glueing data for schemes.

We thus obtain a scheme $\mathbb{P}^r$, which is integral and of finite type over $\mathbb{Z}$ by construction.

The scheme $\mathbb{P}^r$ carries a canonical line bundle $\mathcal{O}(1)$.

Declare $\mathcal{O}(1)|_{U_i} = \mathcal{O}_{U_i}$ and let $\phi_{ij} \in \Gamma(U_{ij}, \mathcal{O}_{U_{ij}})^* = R_{ij}^*$ be given by X_i/X_j. We verify that

$$\phi_{ii} = 1$$

$$\phi_{ij} = \phi_{ji}^{-1}$$

and

$$\frac{X_j}{X_k} \cdot \frac{X_i}{X_j} = \frac{X_i}{X_k}$$

in $R_{i,\frac{X_j}{X_i}\cdot\frac{X_k}{X_i}}$, so that the ϕ_{ij} satisfy the glueing conditions for sheaves.

We thus obtain an abelian sheaf on $\mathbb{P}^r$.

By construction, $\mathcal{O}(1)$ is a quasi-coherent locally free sheaf of rank one.

For each $l = 0, \ldots, r$, there is a canonical element $X_l \in \Gamma(\mathbb{P}^r, \mathcal{O}(1))$, such that

$$X_l|_{U_i} = X_l/X_i$$

via the identification $\mathcal{O}(1)|_{U_i} = \mathcal{O}_{U_i}$. This defines an element of $\Gamma(\mathbb{P}^r, \mathcal{O}(1))$, because

$$\phi_{ij}((X_l|_{U_i})|_{U_j}) = (X_l/X_i) \cdot (X_i/X_j) = X_l/X_j = (X_l|_{U_j})|_{U_i}.$$

so that the local sections $X_l|_{U_i}$ glue to a global section of $\mathcal{O}(1)$.

Since $X_l|_{U_l}$ is a trivialisation of $\mathcal{O}(1)|_{U_l}$, we see that the collection of the X_l defines a surjection

$$\bigoplus_{k=0}^{r} \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}(1).$$

[EL9]

We shall now show that $\mathbb{P}^r$ represents $\underline{\mathbb{P}}^r$.

Let S be a scheme.

If we are given a morphism $\phi : S \to \mathbb{P}^r$, we obtain by pull-back a surjection

$$\bigoplus_{k=0}^{r} \mathcal{O}_S \to \phi^*(\mathcal{O}(1)).$$

This construction provides a map $\mathbb{P}^r(S) \to \underline{\mathbb{P}}^r(S)$.

We wish to construct an inverse map $\underline{\mathbb{P}}^r(S) \to \mathbb{P}^r(S)$.

So let S be a scheme and let

$$\phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}$$

be a surjection of sheaves, where $\mathcal{L}$ is locally free of rank 1.

We shall call $\sigma_0, \dots, \sigma_r$ the corresponding elements of $\Gamma(S, \mathcal{L})$. Let

$$S_{\sigma_i} := \{s \in S \,|\, \sigma_i \notin \mathfrak{m}_s \cdot \mathcal{L}_s\}$$

The set S_{σ_i} is open because $\mathcal{L}$ is locally free.

By Nakayama's lemma, the section $\sigma_i|_{S_{\sigma_i}}$ induces an isomorphism $\mathcal{O}_{S_{\sigma_i}} \simeq \mathcal{L}|_{S_{\sigma_i}}$.

Identifying $\mathcal{L}|_{S_{\sigma_i}}$ with $\mathcal{O}_{S_{\sigma_i}}$ via this isomorphism, we obtain by restriction a morphism

$$\phi_{S_{\sigma_i}} : \bigoplus_{k=0}^{r} \mathcal{O}_{S_{\sigma_i}} \to \mathcal{O}_{S_{\sigma_i}}$$

whose l-th component is given σ_l/σ_i, where it is understood that σ_l/σ_i is a function on S_{σ_i} such that

$$(\sigma_l/\sigma_i) \cdot \sigma_i|_{S_{\sigma_i}} = \sigma_l|_{S_{\sigma_i}}.$$

By Proposition 6.1, $\phi_{S_{\sigma_i}}$ induces a morphism $f_i : S_{\sigma_i} \to U_i$, such that

$$(X_l/X_i) \circ f_i = \sigma_l/\sigma_i.$$

Now note that by construction, we have

$$f_i^{-1}(U_{ij}) = S_{\sigma_i} \cap S_{\sigma_j}$$

and similarly

$$f_j^{-1}(U_{ji}) = S_{\sigma_i} \cap S_{\sigma_j}.$$

Let $\psi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ be the canonical isomorphism (which is the identity in the above presentation).

We compare $\psi_{ij} \circ f_i|_{S_{\sigma_i} \cap S_{\sigma_j}}$ and $f_j|_{S_{\sigma_i} \cap S_{\sigma_j}}$. We compute

$$f_j|^*_{S_{\sigma_i} \cap S_{\sigma_j}}(X_l/X_j) = \sigma_l/\sigma_j$$

and

$$\begin{aligned} & \psi_{ij} \circ f_i|^*_{S_{\sigma_i} \cap S_{\sigma_j}}(X_l/X_j) = \psi_{ij} \circ f_i|^*_{S_{\sigma_i} \cap S_{\sigma_j}}((X_l/X_i) \cdot (X_j/X_i)^{-1}) \\ = \ & (\sigma_l/\sigma_i) \cdot (\sigma_j/\sigma_i)^{-1} = \sigma_l/\sigma_j \end{aligned}$$

so that $\psi_{ij} \circ f_i|_{S_{\sigma_i} \cap S_{\sigma_j}} = f_j|_{S_{\sigma_i} \cap S_{\sigma_j}}$.

Thus the family (f_i) of morphisms glue to a morphism $S \to \mathbb{P}^r$ and we have produced an inverse map $\underline{\mathbb{P}}^r(S) \to \mathbb{P}^r(S)$. QED

Ample line bundles.

Let S be a noetherian scheme.

A coherent F on S is said to be *generated by its global sections* or *globally generated* if there is a surjection

$$\bigoplus_{k=1}^{r_0} \mathcal{O}_S \to F$$

for some $r_0 \in \mathbb{N}$.

The corresponding r_0 sections of F are then called *generating sections.*

Let now L be a line bundle on S.

Definition

The line bundle L is ample if for any coherent sheaf F on S, there is $n_0 \in \mathbb{N}$ such that $F \otimes L^{\otimes n}$ is generated by its global sections for all $n \geqslant n_0$.

Proposition

The line bundle L is ample if and only if there is $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_k \in \Gamma(S, L^{\otimes n})$ such that

- *the schemes S_{σ_i} are affine;*
- *the schemes S_{σ_i} cover S.*

For the **proof**, we shall need the following

Lemma

Let T_0 be a noetherian scheme and let M_0 be a coherent sheaf on T_0. Let L_0 be a line bundle on T_0. Let $f \in \Gamma(T_0, L_0)$ and let $s \in \Gamma(T_{0,f}, M_0)$. Then

(a) *there is $n(s) \in \mathbb{N}$ such that $s \otimes f^{\otimes n(s)} \in \Gamma(T_{0,f}, M_0 \otimes L_0^{n(s)})$ extends to $\Gamma(T_0, M_0 \otimes L_0^{n(s)})$;*

(b) *if $s \in \Gamma(T_0, M_0)$ restricts to 0 in $\Gamma(T_{0,f}, M_0)$ then there is $n(s) \in \mathbb{N}$ such that $s \otimes f^{\otimes n(s)} \in \Gamma(T_0, M_0 \otimes L_0^{n(s)})$ vanishes.*

See Exercises.

Proof (of the last Proposition).

"$\Leftarrow$".

So suppose that there is $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_k \in \Gamma(S, L^{\otimes n})$ such that (S_{σ_i}) is an open affine covering of S.

Let F be a coherent sheaf on S.

For each i, let $(\tau_{ij} \in \Gamma(S_{\sigma_i}, F|_{S_{\sigma_i}})$ be a finite family of generating sections of $F|_{S_{\sigma_i}}$. Such sections exist because S_{σ_i} is affine.

By the last Lemma, there is $n \in \mathbb{N}$ such that for all i, the sections $\tau_{ij} \otimes \sigma_i^{\otimes n}|_{S_{\sigma_i}}$ extend to sections $\lambda_{ij} \in \Gamma(S, F \otimes L^{\otimes n})$.

Now notice that the sections $\tau_{ij} \otimes \sigma_i^{\otimes n}|_{S_{\sigma_i}}$ are also generating sections of $F \otimes L^{\otimes n}|_{S_{\sigma_i}}$ because $L|_{S_{\sigma_i}}$ is by construction trivial.

Hence the sections λ_{ij} (for all i, j) are generating sections of $\Gamma(S, F \otimes L^{\otimes n})$, since the S_{σ_i} cover S.

"$\Rightarrow$":

Let $x \in S$. It is sufficient to show that there is $n(x) \in \mathbb{N}$ and $\sigma_x \in \Gamma(S, L^{\otimes n(x)})$ such that S_{σ_x} is affine and $x \in S_{\sigma_x}$.

Let U be an affine neighbourhood of x such that $L|_U \simeq \mathcal{O}_U$ and let I be the ideal sheaf associated with $S \backslash U$.

Let $\iota : (S \backslash U)_{\text{red}} \to S$ be the canonical closed immersion.

Let $n(x) \in \mathbb{N}$ be such that there is $\bar{\sigma}_x \in \Gamma(S, I \otimes L^{\otimes n(x)})$ with $\bar{\sigma}_x \neq 0$.

Now consider the sequence of $\mathcal{O}_S$-modules

$$0 \to I \to \mathcal{O}_S \to \iota_*(\mathcal{O}_{(S\setminus U)_{\mathrm{red}}}) \to 0 \tag{5}$$

and the sequence

$$0 \to I \otimes L^{\otimes n(x)} \to L^{\otimes n(x)} \to \iota_*(\mathcal{O}_{(S\setminus U)_{\mathrm{red}}}) \otimes L^{\otimes n(x)} \to 0 \tag{6}$$

obtained by tensoring (5) by $L^{\otimes n(x)}$.

Applying $\Gamma(S, \bullet)$ to (6) we obtain a map

$$\Gamma(S, I \otimes L^{\otimes n(x)}) \to \Gamma(S, L^{\otimes n(x)}).$$

Let σ_x be the image of $\bar{\sigma}_x$ by this map.

The section $\sigma_x \in \Gamma(S, L^{\otimes n(x)}$ vanishes on $S\setminus U$ by construction. Hence $S_{\sigma_x} \subseteq U$.

Furthermore, since by assumption we have $L_U \simeq \mathcal{O}_U$, the set $S_{\sigma_x} \subseteq U$ is a basic open subset of the affine scheme U is thus also affine. QED

Corollary

The line bundle $\mathcal{O}(1)$ on $\mathbb{P}^r$ is ample.

Proof. Let X_i be the usual canonical section of $\mathcal{O}(1)$. The schemes $\mathbb{P}^r_{X_i}$ are by construction the affine scheme U_i in the standard open affine covering of $\mathbb{P}^r$. QED

Proposition

Let $f : S \to \operatorname{Spec}(R)$ be a morphism of finite type to the spectrum of a noetherian ring R.

Let L be an ample line bundle on S.

There is $n \in \mathbb{N}$ and $\sigma_0, \ldots, \sigma_r \in \Gamma(S, L^{\otimes n})$ generating $L^{\otimes n}$ and such that the corresponding morphism

$$S \to \mathbb{P}^r_R$$

is a closed immersion into an open subset of $\mathbb{P}^r_R$.

Proof. We may wrog replace L by $L^{\otimes n}$ for some $n \geqslant 1$.

By the above Proposition, we may thus assume that there is a finite family $(\sigma_i \in \Gamma(S, L))$ such that S_{σ_i} is affine and such that the S_{σ_i} cover S.

For each i, let $\sigma_{ij} \in \Gamma(S_{\sigma_i}, L)$ be a family of sections, such that the functions $\sigma_{ij}/\sigma_i|_{S_{\sigma_i}}$ generate $\Gamma(S_{\sigma_i}, \mathcal{O}_{S_{\sigma_i}})$ as an R-algebra.

For some $n \geqslant 0$, which can be taken independent of i, the sections $\sigma_i^{\otimes(n-1)}|_{S_{\sigma_i}} \otimes \sigma_{ij} \in \Gamma(S_{\sigma_i}, L^{\otimes n})$ extend to sections τ_{ij} of $L^{\otimes n}$ over S by Lemma 16.

Now consider the disjoint union Σ of all the σ_i and all the τ_{ij} and choose an arbitrary identification $\phi : \{0, \dots, r\} \simeq \Sigma$.

Since the σ_i already generate L, the set of sections Σ generates $L^{\otimes n}$ and via ϕ we obtain a Spec(R)-morphism $\iota : S \to \mathbb{P}^r_R$.

This morphism is obtained by glueing together the morphisms

$$\iota_i : S_{\sigma_i} \to \mathrm{Spec}(R[\frac{X_0}{X_{\phi^{-1}(\sigma_i)}}, \dots \frac{X_r}{X_{\phi^{-1}(\sigma_i)}}])$$

such that

$$\iota_i^*(\frac{X_k}{X_{\phi^{-1}(\sigma_i)}}) = \frac{\phi(k)|_{S_{\sigma_i}}}{\sigma_i|_{S_{\sigma_i}}}.$$

Since by construction the functions $\frac{\phi(k)|_{S_{\sigma_i}}}{\sigma_i|_{S_{\sigma_i}}}$ generate $\Gamma(S_{\sigma_i}, \mathcal{O}_{S_{\sigma_i}})$ as an R-algebra, and since S_{σ_i} is affine, we see that ι_i is a closed immersion.

Thus ι is a closed immersion of S into the union in $\mathbb{P}^r_R$ of all the open affine subschemes $\mathrm{Spec}(R[\frac{X_0}{X_{\phi^{-1}(\sigma_i)}}, \dots \frac{X_r}{X_{\phi^{-1}(\sigma_i)}}])$ (for all i).

QED

The cohomology of projective space.

If $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^r}(1)$ is the canonical bundle on $\mathbb{P}^r$ and $n \geqslant 0$, we shall write $\mathcal{O}(n)$ for $\mathcal{O}(1)^{\otimes n}$.

For $n < 0$, we also write

$$\mathcal{O}(n) := \mathcal{H}om(\mathcal{O}(-n), \mathcal{O}_{\mathbb{P}^r}) =: (\mathcal{O}(-n))^{\vee}$$

In general, if F is a locally free sheaf on a scheme X, we write

$$F^{\vee} := \mathcal{H}om(F, \mathcal{O}_X)$$

Suppose that L is a locally free sheaf of rank 1 on an integral scheme X.

If $\sigma \in L(X)$ and $\sigma \neq 0$, then σ induces a morphism of sheaves

$$\sigma^\vee : L^\vee \to \mathcal{O}_X$$

whose image is a quasi-coherent sheaf of ideals $\mathcal{I}$.

Since X is integral, the morphism $\sigma^\vee$ is a monomorphism and hence identifies $\mathcal{I}$ with $L^\vee$.

If we let $\iota : Z(\sigma) \to X$ be the closed subscheme associated with $\mathcal{I}$, we thus have an exact sequence

$$0 \to L^\vee \to \mathcal{O}_X \to \iota_*(\mathcal{O}_{Z(\sigma)}) \to 0$$

Note the following simple fact.

If $f : X_0 \to X$ is a morphism of schemes, then there is a morphism $g : X_0 \to Z(\sigma)$ such that $f = \iota \circ g$ iff $f^*(\sigma^\vee) = 0$.

Furthermore, the morphism g, if it exists, is then unique.

From this we deduce $Z(\sigma)$ represents the functor **Schemes** $\mapsto$ **Sets**

$$S \mapsto \{f \in X(S) \mid f^*(\sigma^\vee) = 0\}$$

The closed subscheme $Z(\sigma)$ is called the *zero-scheme* associated with σ.

Lemma

Let $f : X \to Y$ be an affine morphism of schemes.

Suppose that X is noetherian.

Then for all quasi-coherent sheaves F on X, we have $R^k f_*(F) = 0$ for all $k > 0$.

Proof. We may suppose that Y (and thus X) is affine. Now the lemma follows from the fact that $f_* : \mathfrak{Q}coh(X) \to \mathfrak{Q}coh(Y)$ is an exact functor and from the fact that that injective $\mathcal{O}_X(X)$-modules are flasque. QED

Lemma

Let $\iota : X \to Y$ be a closed immersion. Then ι is an affine morphism.

Proof. We may suppose that Y is affine. Then X is given by $\mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y)/J))$ for some ideal of $\Gamma(Y, \mathcal{O}_Y)$. QED

Cech cohomology on noetherian schemes.

Let X be a noetherian scheme and suppose that X a finite open covering (U_i) such that any finite intersection of the U_i is affine.

Let F be a quasi-coherent sheaf on X.

Then we have canonically

$$\mathcal{H}^k(\Gamma(X, \underline{C}^\bullet((U_i), F))) \simeq H^k(X, F)$$

This follows from the existence of the Leray spectral sequence and from the above Lemma.

Proposition

Let A be a noetherian ring.

Then for all $n, k \in \mathbb{Z}$, $H^k(\mathbb{P}^r_A, \mathcal{O}(n))$ is a finitely generated A-module.

Furthermore, we have

$$H^0(\mathbb{P}^r_A, \mathcal{O}) \simeq A$$

and

$$H^k(\mathbb{P}^r_A, \mathcal{O}(n)) = 0$$

for all $k \geqslant 1$ and all $n \geqslant 0$.

Sketch of proof.

First note that for any of the canonical sections

$$X_i = X_i \otimes_A 1 \in \Gamma(\mathbb{P}^r_A, \mathcal{O}(1))$$

the zero scheme $Z(X_i)$ is canonically isomorphic to $\mathbb{P}^{r-1}_A$.

Indeed, for any scheme over $\operatorname{Spec}(A)$ we have

$$\mathbb{P}^r_A(S) = \{\text{iso. cl. of surj. mor. } \phi : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L}\}$$

and the $Z(X_i)$ thus represents the functor
Schemes/$\operatorname{Spec}(A) \mapsto$ **Sets**

$$S \mapsto \{\text{iso. cl. of surj. mor. } \phi = \oplus_k \phi_k : \bigoplus_{k=0}^{r} \mathcal{O}_S \to \mathcal{L} \text{ such that } \phi_i = 0\}$$

which is isomorphic to the functor $\mathbb{P}^{r-1}_A(\bullet)$.

Thus we have an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^r} \to \iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}) \to 0$$

and tensoring this sequence with $\mathcal{O}(k)$ ($k \in \mathbb{Z}$), we obtain a sequence

$$0 \to \mathcal{O}(k-1) \to \mathcal{O}(k) \to \iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) \to 0$$

Note that $\iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}(k)) \simeq \iota_*(\mathcal{O}_{\mathbb{P}^{r-1}}) \otimes \mathcal{O}(k)$.

[EL11]

Now consider the associated long exact sequence:

$$
\begin{aligned}
0 \quad &\to \quad H^0(\mathbb{P}^r_A, \mathcal{O}(k-1)) \to H^0(\mathbb{P}^r_A, \mathcal{O}(k)) \to H^0(\mathbb{P}^{r-1}_A, \iota_*(\mathcal{O}(k))) \\
&\to \quad H^1(\mathbb{P}^r_A, \mathcal{O}(k-1)) \to H^1(\mathbb{P}^r_A, \mathcal{O}(k)) \to H^1(\mathbb{P}^{r-1}_A, \iota_*(\mathcal{O}(k))) \to \ldots
\end{aligned}
$$

Now remember that closed immersions are affine by the above Lemma and thus

$$H^i(\mathbb{P}^r_A, \iota_*(\mathcal{O}(k))) \simeq H^i(\mathbb{P}^{r-1}_A, \mathcal{O}(k)))$$

Thus by a double induction on r and k, we see that it is sufficient to prove that $H^0(\mathbb{P}^r_A, \mathcal{O}) = A$ and $H^k(\mathbb{P}^r_A, \mathcal{O}) = 0$ for all $k > 0$.

This is proven in the notes using Cech cohomology. QED

Corollary

The scheme $\mathbb{P}^r$ is not affine.

Proof. If $\mathbb{P}^r$ were affine, then we would have $\mathbb{P}^r \simeq \mathrm{Spec}(\mathbb{Z})$, according to the theorem. QED

Cohomological properties of strongly projective morphisms

A morphism of schemes $f : X \to S$ is called *strongly projective* if there is a factorisation $f = p \circ \iota$, where $\iota : X \to \mathbb{P}^r_S$ is a closed immersion and $p : \mathbb{P}^r_S \to S$ is the natural projection morphism.

Theorem

Let $f : X \to S$ be a strongly projective morphism.

Suppose that S is a noetherian scheme.

Let F be a coherent sheaf on X.

Then for all $k \geqslant 0$, the sheaf $R^k f_*(F)$ is coherent.

Proof. We may assume that $S = \mathrm{Spec}(R)$, where R is a noetherian ring.

We first show the statement in the case where f is a closed immersion.

In that case, since f is affine, we may also assume that $X = \mathrm{Spec}(T)$ is affine.

We then have $R^k f_*(F) = 0$ for all $k > 0$ by the above Lemma and thus we only have to show that $f_*(F)$ is coherent.

We know that $f_*(F)$ is quasi-coherent by Proposition 5.12 and thus we only have to show that $f_*(F)(S)$ is a finitely generated R-module. This is clear.

By the existence of the Leray spectral sequence, we may thus suppose that $X = \mathbb{P}^r_S$ and that f is the natural projection.

Let n_0 be such that $F \otimes \mathcal{O}(n_0) := F(n_0)$ is globally generated.

Noticing that

$$\mathcal{O}(n_0) \otimes \mathcal{O}(-n_0) \simeq \mathcal{O}_{\mathbb{P}^r_S},$$

we obtain a surjection $\bigoplus_{k=0}^{f} \mathcal{O}(-n_0) \to F$ for some f.

Denoting by K the kernel of this morphism, we get an exact sequence

$$0 \to K \to \bigoplus_{k=0}^{f} \mathcal{O}(-n_0) \to F \to 0 \tag{7}$$

Note that K is also a *coherent* sheaf, because $\mathbb{P}^r_S$ is a noetherian scheme.

Notice also that we may compute to cohomology of F using the Cech complex with ordering associated with the standard open covering of $\mathbb{P}^r_S$.

The terms of this complex vanish in degrees $> r$. Thus we know that $R^k f_*(F) = 0$ for all $k > r$.

Now looking at the long exact cohomology for f_* of (7) we obtain a surjection

$$R^r f_*(\bigoplus_{k=0}^{f} \mathcal{O}(-n_0)) \to R^r f_*(F)$$

Thus we see that $R^r f_*(F)$ is coherent, since we know that $R^r f_*(\oplus_{k=0}^{f} \mathcal{O}(-n_0))$ is coherent.

Since F was arbitrary, we deduce that $R^r f_*(K)$ is also coherent.

The long exact cohomology sequence again now shows that we have an exact sequence

$$R^{r-1} f_*(\bigoplus_{k=0}^{f} \mathcal{O}(-n_0)) \to R^{r-1} f_*(F) \to R^r f_*(K)$$

and thus $R^{r-1} f_*(F)$ is also coherent. Thus $R^{r-1} f_*(K)$ is also coherent and we may continue this way to show that $R^k f_*(F)$ is coherent for all k. QED

Theorem (Serre)

Let $f : X \to \operatorname{Spec}(A)$ be a strongly projective morphism, where A is a noetherian ring.

Let L be an ample line bundle.

Let F be a coherent sheaf on X.

Then there is $n_0 \geqslant 0$ such that $R^k f_*(F \otimes L^{\otimes n}) = 0$ for all $n \geqslant n_0$ and all $k > 0$.

Proof. As before, we may thus suppose that $X = \mathbb{P}^r_A$ for some $r \geqslant 0$.

Since $\mathbb{P}^r_A$ has a finite covering by $r+1$ open affine subschemes whose intersections are affine, we have $H^k(\mathbb{P}^r_A, Q) = 0$ for all $k > r$ and any quasi-coherent sheaf Q on $\mathbb{P}^r_A$.

Let n_0 be sufficiently large so that $F \otimes L^{\otimes n_0}$ is generated by its global sections. In other words, we have an exact sequence

$$0 \to K \to \bigoplus_{i=1}^{r_0} \mathcal{O} \to F(n_0) \to 0 \tag{8}$$

Looking at the long exact cohomology sequence of (120), we get a surjection

$$H^r(\mathbb{P}^r_A, \bigoplus_{i=1}^{r_0} \mathcal{O}) \to H^r(\mathbb{P}^r_A, F(n_0)).$$

and thus $H^r(\mathbb{P}^r_A, F(n_0)) = 0$.

Now take n_1 so that $K(n_1)$ is also globally generated. Then we also have $H^r(\mathbb{P}^r_A, K(n_1)) = 0$. Looking at the long exact cohomology sequence of the sequence

$$0 \to K(n_1) \to \bigoplus_{i=1}^{r_0} \mathcal{O}(n_1) \to F(n_0 + n_1) \to 0$$

we get a surjection

$$H^{r-1}(\mathbb{P}^r_A, \bigoplus_{i=1}^{r_0} \mathcal{O}(n_1)) \to H^{r-1}(\mathbb{P}^r_A, F(n_0 + n_1))$$

and again we see that $H^{r-1}(\mathbb{P}^r_A, F(n_0 + n_1)) = 0$, unless $r - 1 = 0$.

Continuing this way, we conclude that $F(n)$ has no cohomology in positive degrees for n sufficiently large.

Cohomological characterisation of ample line bundles.

Theorem

Let X be a noetherian scheme.

Let L be a line bundle on X.

Suppose that for all coherent sheaves F on X, there is $n_0 \geqslant 0$ such that $H^k(X, F \otimes L^{\otimes n}) = 0$ for all $n \geqslant n_0$ and all $k > 0$.

Then L is ample.

The proof is similar to the proof of Serre's theorem characterising affine schemes. See for the notes for details.

[EL12]

Flat morphisms

Let $f : X \to Y$ be a morphism of schemes.

Definition

Let F be a $\mathcal{O}_X$-module. We say that F is flat over Y at $x \in X$ if the stalk F_x is flat as a $\mathcal{O}_{Y,f(x)}$-module via the natural morphism of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. We say that F is flat over Y if F is flat at every $x \in X$.

Recall that a module M over a ring R is flat if the functor $\bullet \otimes M$ from R-modules to R-modules sending an R-module N to $N \otimes M$ is an exact functor.

- Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a morphism schemes and F a quasi-coherent sheaf on $\mathrm{Spec}(B)$. Let M be the B-module associated with F. Then F is flat over $\mathrm{Spec}(A)$ if and only if M is flat as an A-module: see Exercises.

- Any base-change of a flat morphism is flat. See Exercises.

We also recall without proof the following basic result:

Theorem

Let A be a local ring and let M be a finite A-module. Then the following conditions on M are equivalent.

- *M is flat over A;*
- *M is free over A.*

Proof.

See Theorem 7.10 in *Commutative Ring Theory* by H. Matsumura. □

A consequence of this theorem is that a coherent sheaf F on a noetherian scheme X is flat over X if and only if it is locally free.

Cohomology and flat base change

Let $f : X \to Y$ be a morphism of schemes.

Suppose that Y is noetherian and affine and suppose that X has a finite open covering (U_i) of X such that any finite intersection of the U_i is affine.

Let F be a quasi-coherent sheaf on X. Let

$$\begin{array}{ccc} X' & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{b} & Y \end{array}$$

be a cartesian diagram, where Y' is a noetherian and affine and b is flat.

Theorem

There is a natural isomorphism $b^*(R^l f_*(F)) \simeq R^l f'_*(r^*(F))$.

Proof. Consider the complex

$$L^\bullet := f_*(\underline{C}^\bullet((U_i), F)).$$

By construction we have

$$\mathcal{H}^l(b^*(L^\bullet)) \simeq R^l f'_*(r^*(F)).$$

Now since b is flat, we see that $\mathcal{H}^l(b^*(L^\bullet)) \simeq b^*(\mathcal{H}^l(L^\bullet))$, in other words we have

$$b^*(R^l f_*(F)) \simeq R^l f'_*(r^*(F)).$$

QED

The semicontinuity theorem

Let $f : X \to Y$ be a morphism of schemes.

Suppose that Y is noetherian and affine and suppose that X has a finite open covering (U_i) of X such that any finite intersection of the U_i is affine.

Let F be a quasi-coherent sheaf on X and suppose that

- the sheaf F is flat over Y;
- for all $l \geqslant 0$, the quasi-coherent sheaf $R^l f_*(F)$ is coherent.

Theorem (semicontinuity theorem)

There is a finite cochain complex of coherent locally free modules $(K^\bullet)$ *on* Y *with the following property.*

For any cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "r"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"] & Y \end{tikzcd}$$

where Y' *is a noetherian and affine, there is a canonical isomorphism of quasi-coherent sheaves*

$$\mathcal{H}^l(b^*(K^\bullet)) \simeq R^l f'_*(r^*F).$$

Note that the assumptions of the theorem will be verified if f is a strongly projective morphism, F is flat over Y and Y is noetherian and affine.

Sketch of proof.

The proof is a consequence of the existence of the Cech complex and of two basic results of homological algebra.

Step I. Homological algebra.

Let R be a noetherian ring. Let

$$0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

be a finite cochain complex of R-modules.

Suppose that $\mathcal{H}^i(C^\bullet)$ is finitely generated for all $i \geqslant 0$.

Lemma

There is a finite cochain complex of R-modules

$$0 \to L^0 \to L^1 \to \cdots \to L^n \to 0$$

such that

- $L^\bullet$ *is quasi-isomorphic to* $C^\bullet$*;*
- L^i *is free for all* $i > 0$*;*
- L^i *is finitely generated for all* $i \geqslant 0$*.*

Furthermore, C^i *is flat for all* $i \geqslant 0$ *then we may find a cochain complex* $L^\bullet$ *with the above properties such that* L^0 *is flat.*

For the proof, see the notes.

Let $\phi : P_1^\bullet \to P_2^\bullet$ be a quasi-isomorphism of cochain complexes of R-modules, where both complexes are supposed bounded above.

Let $C^\bullet$ be another cochain complex of R-modules, which is bounded above. Suppose that either

- C^k is a flat R-module for all $k \in \mathbb{Z}$
- or P_1^k and P_2^k are flat R-modules for all $k \in \mathbb{Z}$.

Lemma

The morphism

$$\phi \otimes C^\bullet : P_1^\bullet \otimes C^\bullet \to P_2^\bullet \otimes C^\bullet$$

is a quasi-isomorphism.

Proof of the semicontinuity theorem.

Consider the complex

$$L^\bullet := f_*(\underline{C}^\bullet((U_i), F)).$$

Then L^i are flat and quasi-coherent, $L^\bullet$ is a finite complex and by construction for any cartesian diagram as in the statement of the theorem, we have

$$\mathcal{H}^l(b^*(L^\bullet)) \simeq R^l f'_*(r^* F).$$

Hence, by the last lemma, we only have to show that there exists a complex of coherent locally free modules $K^\bullet$, which is quasi-isomorphic to $L^\bullet$.

The lemma before provides this complex. QED

[EL13]

Consequences of the semicontinuity theorem.

If we apply Nakayama's lemma to the complex provided by the semi-continuity theorem, we can derive interesting statements about the cohomology of flat families.

Lemma (Nakayama's lemma)

Let R be a local ring with maximal ideal $\mathfrak{m}$.

Let M be a finite R-module.

Let $b_1, \ldots, b_k \in M$ be pairwise distinct elements.

Then the set $\{b_1, \ldots, b_k\}$ is a set of generators of M of minimal cardinality if and only if the image of $\{b_1, \ldots, b_k\}$ in $M/\mathfrak{m}M$ is a basis of $M/\mathfrak{m}M$ as a $R/\mathfrak{m}$-vector space.

See Atiyah-MacDonald for the proof.

If R is a ring and $L^\bullet$ is a cochain complex of R-modules.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$.

We denote by $L^\bullet_\mathfrak{p}$ the complex on $R_\mathfrak{p}$ obtained by localisation and we write $L^\bullet(\mathfrak{p})$ for the complex $L^\bullet_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, which is a complex of $\kappa(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$-vector spaces.

Lemma

Let R be a noetherian ring. Let M be a finitely generated R-module.

Then the function $\dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p}))$ is upper semicontinuous on $\operatorname{Spec}(R)$, ie for all $n \in \mathbb{Z}$, the set

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p})) \geqslant n\}$$

is closed.

If R is reduced and $\dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p}))$ is constant then M is locally free.

Proof of the first assertion. Let

$$R^t \to R^s \to M \to 0$$

be an exact sequence. Let $\mathfrak{p} \in \mathrm{Spec}(R)$.

Let $(\phi_{ij})_{1 \leqslant i \leqslant s; 1 \leqslant i \leqslant t}$ be a $s \times t$ matrix representing the map $R^t \to R^s$ in the standard bases.

For each $l \geqslant 1$, let $f_{1l}, \ldots f_{k_l l}$ be the set of all the minors of order l of (ϕ_{ij}) (these are polynomials in the ϕ_{ij}). We then have

$$\begin{aligned}
& \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, \dim_{\kappa(\mathfrak{p})}(M(\mathfrak{p})) \geqslant n\} \\
= \;& \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, s - \mathrm{rk}((\phi_{ij}(\mathfrak{p}))) \geqslant n\} \\
= \;& \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, \mathrm{rk}((\phi_{ij}(\mathfrak{p}))) \leqslant s - n\} \\
= \;& \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, \forall l > s-n,\ r \geqslant 1 : f_{lr} \in \mathfrak{p}\} \\
= \;& \cap_{l > s-n} \cap_{r=1}^{k_l} V((f_{lr}))
\end{aligned}$$

proving the first assertion in the lemma.

Proof of the second assertion. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and let $\gamma_1, \ldots, \gamma_r$ be a basis of $M(\mathfrak{p})$. We have to show that $\widetilde{M}$ is locally free in a neighborhood of $\mathfrak{p}$.

Lift this basis to a set $a_1/b_1, \ldots, a_r/b_r \in M_{\mathfrak{p}}$, where $b_1, \ldots b_r \in R \backslash \mathfrak{p}$. We may and do replace R by $R_{b_1 \cdots b_r}$, since $R_{b_1 \cdots b_r}$ corresponds to a basic open set of R.

Consider now the exact sequence of R-modules

$$0 \to K \to R^r \xrightarrow{\phi} M \to C \to 0 \tag{9}$$

where $\phi((x_1, \ldots, x_r)) = \sum_i x_i \cdot \frac{a_i}{b_i}$.

By construction $C(\mathfrak{p}) = 0$ and by Nakayama's lemma, we conclude that $C_{\mathfrak{p}} = 0$.

Since C is a finitely generated R-module, this means that there exists $b \in R \backslash \mathfrak{p}$ such that $b \cdot C = 0$ and thus replacing again R by R_b, we obtain a sequence of R-modules

$$0 \to K \to R^r \xrightarrow{\phi} M \to 0$$

Now K is a finitely generated R-module as well, since R is noetherian and we choose a surjection $R^t \to K$. This yields another exact sequence of R-modules

$$R^t \xrightarrow{\lambda} R^r \xrightarrow{\phi} M \to 0$$

Now since $\dim_{\kappa(\mathfrak{q})}(M(\mathfrak{q})) = r$ for all $\mathfrak{q} \in \mathrm{Spec}(R)$, we see that $\phi(\mathfrak{q})$ is an isomorphism and $\lambda(\mathfrak{q}) = 0$ for all $\mathfrak{q} \in \mathrm{Spec}(R)$.

The map λ can be described by a matrix $(\psi_{ij} \in R)$ and we have just shown that for all i, j and all $\mathfrak{q} \in \mathrm{Spec}(R)$, we have $\psi_{ij}(\mathfrak{q}) = 0$.

In other words, $\psi_{ij} \in \sqrt{(0)} = 0$ for all i, j and thus $\psi_{ij} = 0$ and $\lambda = 0$. Thus ϕ is an isomorphism and M is free. QED

Lemma

Let R be a reduced noetherian ring. Let

$$0 \to L^0 \xrightarrow{d^0} L^1 \xrightarrow{d^1} \cdots \to L^n \to 0$$

be a finite cochain complex of finitely generated free R-modules. Suppose that the function on $\operatorname{Spec}(R)$

$$\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})}(H^i(L^\bullet(\mathfrak{p})))$$

is constant. Then $H^i(L^\bullet)$ is free and there is a natural isomorphism

$$H^i(L^\bullet)(\mathfrak{p}) \simeq H^i(L^\bullet(\mathfrak{p}))$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

See the notes for the proof, which relies on Nakayama's lemma.

Lemma

Let R be a noetherian ring. Let

$$0 \to L^0 \to L^1 \to \cdots \to L^n \to 0$$

be a finite cochain complex of finitely generated free R-modules. Then the function on $\mathrm{Spec}(R)$

$$\mathfrak{p} \mapsto \sum_{i \geqslant 0} (-1)^i \dim_{\kappa(\mathfrak{p})}(H^i(L^\bullet(\mathfrak{p})))$$

is locally constant on R.

Proof. Notice that

$$\sum_{i \geqslant 0} (-1)^i \dim_{\kappa(\mathfrak{p})}(H^i(L^\bullet(\mathfrak{p}))) = \sum_{i \geqslant 0} (-1)^i \dim_{\kappa(\mathfrak{p})}(L^\bullet(\mathfrak{p})) = \sum_{i \geqslant 0} (-1)^i \mathrm{rk}(I$$

We leave it as an exercise to check this (hint: lift bases). QED

Corollary

Let $f : X \to Y$ be a strongly projective morphism. Suppose that Y is noetherian.

Let F be a coherent sheaf on X and suppose that F is flat over Y.

Then the function

$$y \mapsto \sum_{i \geqslant 0} (-1)^i \dim_{\kappa(y)}(H^i(X_y, F_y))$$

is locally constant on Y.

Proof. Apply the last Lemma to the complex $K^\bullet$ provided by the semicontinuity theorem. QED

Corollary

Let $f : X \to Y$ be a strongly projective morphism. Suppose that Y is noetherian and reduced.

Let F be a coherent sheaf on X and suppose that F is flat over Y.

Suppose that the function

$$y \mapsto \dim_{\kappa(y)}(H^i(X_y, F_y))$$

is locally constant on Y.

Then $R^i f_*(F)$ is locally free.

Proof. Apply the Lemma before last to the complex $K^\bullet$ provided by the semicontinuity theorem. QED

Hilbert polynomials

Let $r \geqslant 0$ and let K be a field.

Let F be a coherent sheaf on $\mathbb{P}^r_K$.

For all $n \in \mathbb{Z}$, we write

$$\chi_F(n) := \sum_{i \geqslant 0} (-1)^i \dim_K H^i(X, F \otimes \mathcal{O}(n))$$

Proposition

The function $\chi_F(\bullet)$ is a polynomial with rational coefficients.

The polynomial $\chi_F(\bullet)$ is called the *Hilbert polynomial* of F.

Example

We have

$$\chi\mathcal{O}_{\mathbb{P}^r_K}(n) = \binom{n+r}{n} = \dim_K K[X_0, \ldots, X_r]^{[n]}$$

For the proof of the proposition, see the notes.

Note also the important

Lemma

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of coherent sheaves on $\mathbb{P}^r_K$*, then we have*

$$\chi_F(n) = \chi_{F'}(n) + \chi_{F''}(n)$$

for all $n \in \mathbb{Z}$*.*

Proof (of the lemma). Look at the associated long exact sequence of cohomology. QED

Proposition

Let S be a connected locally noetherian scheme.

Let $r \geqslant 0$ and let $\iota : X \to \mathbb{P}^r_S$ be a closed subscheme of $\mathbb{P}^r_S$.

Suppose that X is flat over $\mathrm{Spec}(A)$.

Then the Hilbert polynomial of $\iota_{\kappa(s)} : X_{\kappa(s)} \to \mathbb{P}^r_{\kappa(s)}$ does not depend on $\mathfrak{p} \in S$.

Here the immersion $\iota_{\kappa(s)} : X_{\kappa(s)} \to \mathbb{P}^r_{\kappa(s)}$ is obtained by base-change from $\iota : X \to \mathbb{P}^r_S$ via the natural morphism $\mathrm{Spec}(\kappa(s)) \to S$.

Proof. This is a special case of a corollary of the semicontinuity theorem. QED

Two more results on flatness. (without proof; not in the notes)

Theorem (generic flatness theorem)

Let $f : X \to Y$ be a morphism of finite type.

Suppose that Y is noetherian and integral.

Let F be a coherent sheaf on X.

Then there is a non-empty open set $U \subseteq Y$ such that the restriction of F to $f^{-1}(U)$ is flat over U.

Theorem (numerical characterisation of flatness)

Let S be an integral noetherian scheme, let $r \geqslant 0$ and let $\iota : X \to \mathbb{P}^r_S$ be a closed subscheme of $\mathbb{P}^r_S$.

Suppose that the Hilbert polynomial of $\iota_{\kappa(s)} : X_{\kappa(s)} \to \mathbb{P}^r_{\kappa(s)}$ does not depend on $\mathfrak{p} \in S$.

Further results on the Zariski topology

Definition

Let T be a topological space. We say that T is irreducible if every non empty open subset of T is dense in T.

Equivalently, T is irreducible iff there is no pair of disjoint non empty open subsets in T.

Notice that every open subset of an irreducible topological space is irreducible.

Lemma

If A is a noetherian ring then $\mathrm{Spec}(A)$ is irreducible if and only if $A/\sqrt{(0)}$ is an integral ring.

Corollary

Let A be a noetherian ring. Let $I \subseteq A$ be an ideal.

Then $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.

Lemma

Let T be a noetherian topological space.

There is a finite sequence $C_1, \ldots C_k$ of closed irreducible subsets of T such that

- $\bigcup_i C_i = T$;
- *for all indices i, we have $C_i \not\subseteq \cup_{j \neq i} C_j$.*

This sequence is unique up to permutation of the indices.

See Exercises.

Application. Let A be a noetherian ring. There is a finite sequence $\mathfrak{p}_1, \dots \mathfrak{p}_k$ of prime ideals in A such that

- $\bigcap_i \mathfrak{p}_i = \sqrt{0}$;
- for all indices i, we have $\mathfrak{p}_i \not\supseteq \cap_{j \neq i} \mathfrak{p}_j$.

This sequence is unique up to permutation of the indices. The ideals $\mathfrak{p}_1, \dots \mathfrak{p}_k$ are called the *minimal prime ideals* of A.

In particular, if $I \subseteq A$ is an ideal, there is a finite sequence $\mathfrak{p}_1, \dots \mathfrak{p}_k$ of prime ideals in A such that

- $\bigcap_i \mathfrak{p}_i = \sqrt{I}$;
- for all indices i, we have $\mathfrak{p}_i \not\supseteq \cap_{j \neq i} \mathfrak{p}_j$.

This sequence is unique up to permutation of the indices.

The following lemma points out a specific property of irreducible closed subsets of schemes.

Lemma (generic points)

Let S be a scheme. Let $C \subseteq S$ be an irreducible closed subset.

There is a unique point $\eta \in C$ such that the Zariski closure $\bar{\eta}$ is C.

The point η is called *the generic point* of C.

Idea of proof. Reduce to $S = C$ and S the spectrum of an integral ring A. Then the generic point is given by (0).

Constructibility.

Definition

Let T be a noetherian topological space.

A subset $E \subseteq T$ is called constructible if E is a finite union of locally closed subsets.

The class of constructible sets is the smallest subclass of the power set of T, which contains the open subsets of T and is closed under finite unions and complementation.

Noetherian induction.

Let T be a noetherian topological space.

Let $P(\bullet)$ be a property of closed subsets of T.

Suppose that $P(\emptyset)$ holds and that for all closed subsets C of T, the statement

$$\textit{if } P(C') \textit{ holds for all closed subsets } C' \overset{\neq}{\hookrightarrow} C \textit{ then } P(C) \textit{ holds}$$

is verified.

Then $P(T)$ holds.

Lemma

Let T be a noetherian topological space.

Then $E \subseteq T$ is constructible if and only if for any irreducible closed subset $C \subseteq T$, either $E \cap C$ or $C \setminus (E \cap C)$ contains a non empty open subset of C in the induced topology.

Proof of "$\Rightarrow$". By noetherian induction, we may assume that $T = C$ and thus that T is irreducible. So we have to show that either E or $T \backslash E$ contains an open subset of T.

Let $T_1, \dots T_k$ be closed subsets of T and $U_1 \subseteq T_1, \dots, U_k \subseteq T_k$, where for all indices i, the set U_i is an open subset of T_i in the induced topology. Suppose that $E = \bigcup_i U_i$.

If for some index i_0, we have $T_{i_0} = T$, then E contains an open subset of T. So we may suppose that $T_i \neq T$ for all i. Now since T is irreducible, we have $T \neq \bigcup_i T_i$ and thus

$$T \backslash (\bigcup_i T_i) \subseteq T \backslash E \neq \emptyset. \qquad \text{QED}$$

Permanence properties of constructible sets.

Let $f : X \to Y$ be a morphism of noetherian schemes. Let $E \subseteq Y$ be constructible.

Then $f^{-1}(E)$ is clearly constructible.

What about direct images?

Theorem (Chevalley-Tarski)

Let $f : X \to Y$ be a morphism of finite type. Suppose that Y is noetherian.

Let $E \subseteq X$ be a constructible subset of X.

Then $f(E)$ is a constructible subset of Y.

[EL15]

Main steps of the proof of Chevalley-Tarski's theorem.

Step I. Preliminary results in commutative algebra.

Theorem (Noether's normalisation lemma)

Let K be a field and let A be a finitely generated K-algebra. Then there is a natural number $n \in \mathbb{N}$ and a map of K-algebras

$$\phi : K[T_1, \dots T_n] \to A$$

such that ϕ is injective and finite.

By definition, ϕ is *finite* if A is a finitely generated $K[T_1, \dots T_n]$-module.

Theorem (going-up theorem)

Let $\phi : A \to B$ be a morphism of rings and suppose that ϕ is injective and finite. Then $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Proof of the Going up theorem.

Lemma

Suppose that $\lambda : k \to B_0$ is an injective and finite map of domains. Then B_0 is a field if and only if k is a field.

Proof (of Lemma 31).

"$\Rightarrow$". Suppose that k is a field.

By induction on the number of generators of B_0 as a k-module, we may suppose that B_0 is generated by one element $b_0 \in B_0$ over k.

Let $k[t] \to B_0$ be the k-algebra map sending t on b_0.

The kernel of this map is a prime ideal, since B_0 is integral.

Since prime ideals in $k[t]$ are maximal, we conclude that B_0 is a field.

"$\Leftarrow$". Now suppose that B_0 is a field. We want to show that k is a field.

Let $x \in k^*$. We only have to show that the inverse $x^{-1} \in B_0$ lies in k.

Let $e_x : B_0 \to B_0$ be the map such that $e_x(z) = z/x$ for all $z \in B_0$.

There is a polynomial $P(t) = t^n + a_{n-1} \cdot t^{n-1} + \cdots + a_0 \in k[t]$ such that $P(e_x) = 0$ (generalised Cayley-Hamilton).

In particular, we have $P(e_x)(1) = P(1/x) = 0$.

Thus we have $x^{n-1} \cdot P(1/x) = 0$, ie

$$x^{-1} + a_{n-1}x + \cdots + a_0 \cdot x^{n-1} = 0$$

which implies that $x^{-1} \in k$. QED

End of proof of the Going-up theorem.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$. There is a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(B_{\mathfrak{p}}) \arrow[r] \arrow[d, "\mathrm{Spec}(\phi_{\mathfrak{p}})"] & \mathrm{Spec}(B) \arrow[d, "\mathrm{Spec}(\phi)"] \\ \mathrm{Spec}(A_{\mathfrak{p}}) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Since $\mathfrak{p}$ the image of the maximal ideal $\mathfrak{m}$ of $A_{\mathfrak{p}}$ under the map $\mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$, it is sufficient to show that there is a prime ideal $\mathfrak{q}$ in $B_{\mathfrak{p}}$ so that $\phi_{\mathfrak{p}}^{-1}(\mathfrak{q}) = \mathfrak{m}$.

Let $\mathfrak{q}$ be any maximal ideal of $B_{\mathfrak{p}}$.

We have an injective and finite map $A_{\mathfrak{p}}/\phi_{\mathfrak{p}}^{-1}(\mathfrak{q}) \to B_{\mathfrak{p}}/\mathfrak{q}$.

By assumption, the ring $B_{\mathfrak{p}}/\mathfrak{q}$ is a field and by Lemma 31, the ring $A_{\mathfrak{p}}/\phi_{\mathfrak{p}}^{-1}(\mathfrak{q})$ is also field, ie $\phi_{\mathfrak{p}}^{-1}(\mathfrak{q})$ is a maximal ideal in $A_{\mathfrak{p}}$.

Since $A_{\mathfrak{p}}$ is a local ring, we have $\mathfrak{p} = \phi_{\mathfrak{p}}^{-1}(\mathfrak{q})$. QED

Lemma (expanded normalisation lemma)

Let $\phi : A_0 \to B_0$ be an injective morphism of rings.

Suppose that A_0 and B_0 are integral rings and suppose that B_0 is finitely generated as an A_0-algebra.

Then there is $n \in \mathbb{N}$, $s \in A_0$ and a finite and injective homomorphism of A_0-algebras

$$A_{0,s}[t_1, \ldots, t_n] \to B_{0,s}$$

See the notes for the proof.

Proof. (of the theorem of Chevalley-Tarski).

Step I. *If $f(E)$ is Zariski dense in Y then $f(E)$ contains a non empty open subset of Y.*

By noetherian induction, we may assume that $\bar{E} = X$ and assume that the statement hold if $\bar{E} \neq X$.

We may suppose that X and Y are reduced and irreducible (easy).

We may wrog replace Y by one of its open affine subschemes V and X by $X \times_Y V$, so we may assume that Y is affine.

Let now $U \subseteq X$ be a non empty open affine subscheme. Either $f(U \cap E)$ is Zariski dense in Y or $f((X \backslash U) \cap E)$ is Zariski dense in Y. In the latter case, the assertion follows from the noetherian inductive hypothesis so we may assume that $f(U \cap E)$ is Zariski dense in Y and thus replace X by U.

E is a finite union of locally closed subsets and one of these closed subsets, say E_0, must be dense in X.

In particular, E_0 contain an affine open subset U_0 of X and as before, we may replace X by U_0 so that we now have $E = X$.

By Lemma 27, we may thus assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, where A and B are integral rings. Let $\phi : A \to B$ be the corresponding maps of rings. Since $f(X)$ is dense in Y, ϕ is injective.

So we are now reduced to show that that if $\phi : A \to B$ is an injective map of rings, which makes B a finitely generated A-algebra, then $\mathrm{Spec}(\phi)(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$ contains an open subset of $\mathrm{Spec}(A)$.

Now recall that there is $n \in \mathbb{N}$, $s \in A$ and a finite and injective homomorphism of A_0-algebras

$$A_s[t_1, \ldots, t_n] \to B_s$$

We may replace A by A_s and B by B_s, since $\mathrm{Spec}(A_s)$ is a basic open subset of $\mathrm{Spec}(A)$.

In this situation, the going-up theorem implies that $\mathrm{Spec}(\phi)$ is surjective and we have proven the statement and completed Step I.

Step II. **End of proof of Chevalley-Tarski.**

By noetherian induction, we may assume that the Zariski closure of $f(E)$ is Y and that the intersection of $f(E)$ with any proper closed subset of Y is constructible.

Let C be an irreducible closed subset of Y. By Lemma 30, it is sufficient to show that $C \cap f(E)$ or $C \backslash (C \cap f(E))$ contains a non empty open subset of C.

If $C \neq Y$ then by the inductive hypothesis, we know that $C \cap f(E)$ is constructible and in particular $C \cap f(E)$ or $C \backslash (C \cap f(E))$ contains a non empty open subset of C.

So we may assume that $C = Y$. In that case, C contains a non empty open subset by Step I. QED

[EL16]