Solutions for Exercise sheet 4*

(for the last class, at the beginning of Trinity Term)

Exercise 1.1. *Let $\phi : A \to B$ be a morphism of integral rings. Suppose that $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has dense image. Show that ϕ is injective.*

Suppose that ϕ is not injective. Let I be the kernel of ϕ. Since B is a domain and since A/I is isomorphic to a subring of B (by the first isomorphism theorem), we see that I is a prime ideal, in particular a radical ideal. Furthermore, since every ideal contains the 0 element, we see that for any prime ideal $\mathfrak{p} \in \mathrm{Spec}\, B$ we have $I = \phi^{-1}(0_B) \subseteq \phi^{-1}(\mathfrak{p}) =: \mathrm{Spec}(\phi)(\mathfrak{p})$. Thus the image of $\mathrm{Spec}(\phi)$ is contained in the closed subset $V(I)$. By assumption $I \neq (0_A)$ and (0_A) is a radical ideal, since A is a domain. Thus by Lemma 2.3 and Lemma 2.1, $V(I) \neq \mathrm{Spec}\, A$. The closure of $\mathrm{Spec}(\phi)$ is contained in $V(I)$, which we have shown to be a proper closed subset of $\mathrm{Spec}\, A$. Thus the image of $\mathrm{Spec}(\phi)$ is not dense in $\mathrm{Spec}\, A$.

Exercise 1.2. *Let X be a noetherian scheme and let L, M be line bundles on X.*

(a) Suppose that L is ample. Show that for sufficiently large $n \geqslant 0$, the line bundle $L^{\otimes n} \otimes M$ is ample.

(b) Suppose that L and M are ample. Show that the line bundle $L \otimes M$ is ample.

(a) Let F be a coherent sheaf on X. Choose $n \geq 0$ so that the sheaf $L^{\otimes(n-1)} \otimes M$ is generated by the sections $a_1, \dots, a_{i_1}$. By the definition of ampleness, this is possible for all sufficiently large $n \geq 0$.

We shall show that there is a $k_0 \geq 0$ such that when $k \geq k_0$, the sheaf

$$(L^{\otimes n} \otimes M)^{\otimes k} \otimes F$$

is generated by some of its sections. By definition of ampleness, this will prove what we want. Note that there are natural isomorphism

$$(L^{\otimes n} \otimes M)^{\otimes k} \otimes F \simeq (L^{\otimes(n-1)} \otimes M)^{\otimes k} \otimes (L^{\otimes k} \otimes F).$$

Choose k_0 so that $L^{\otimes k} \otimes F$ is generated by global sections for all $k \geq k_0$. Fix such a $k \geq k_0$ and let $b_1, \dots, b_{j_1}$ be generating sections of $L^{\otimes k} \otimes F$. For $1 \leq i \leq i_1$ and $1 \leq j \leq j_1$, the sections $a_i^{\otimes k} \otimes b_j$ are then generating sections for $(L^{\otimes(n-1)} \otimes M)^{\otimes k} \otimes (L^{\otimes k} \otimes F)$.

(b) Notice first that if F is generated by the global sections $a_1, \dots, a_{i_1}$ and G is generated by the global sections $b_1, \dots, b_{j_1}$ then $F \otimes G$ is generated by the global sections $a_i \otimes b_j$ (where $1 \leq i \leq i_1$ and $1 \leq j \leq j_1$). Now let F be a coherent sheaf F on X. We have to show that for all sufficiently large n, the sheaf

$$(L \otimes M)^{\otimes n} \otimes F$$

*For the course C2.6 Introduction to Schemes, Oxford, Hilary Term 2019 is generated by global sections. Note that we have a natural isomorphism

$$(L \otimes M)^{\otimes n} \otimes F \simeq L^{\otimes n} \otimes (M^{\otimes n} \otimes F)$$

for any $n \geq 0$. Now if we choose n sufficiently large, the sheaves $L^{\otimes n}$ and $M^{\otimes n} \otimes F$ will be globally generated (since L is ample) and thus by the above remark, so will be $L^{\otimes n} \otimes (M^{\otimes n} \otimes F)$.

Exercise 1.3. *Let L be a line bundle on $\mathbb{P}^1_{\mathbb{C}}$. Let $\sigma \in \Gamma(\mathbb{P}^1_{\mathbb{C}}, L)$ and suppose that $\sigma \neq 0$. Let $Z(\sigma) \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$ be the zero scheme associated with σ. Let $V_\sigma := \Gamma(Z(\sigma), \mathcal{O}_{Z(\sigma)})$ where V_σ is viewed as a $\mathbb{C}$-vector space. Prove that $\dim_{\mathbb{C}}(V_\sigma)$ is independent of σ. It is called the* **degree** *of L.*

Write $X = \mathbb{P}^1_{\mathbb{C}}$. Let $\iota : Z(\sigma) \hookrightarrow X$ be the given closed immersion. We have by definition an exact sequence of coherent sheaves

$$0 \to L^\vee \to \mathcal{O}_X \to \iota_*(\mathcal{O}_{Z(\sigma)}) \to 0$$

and taking into account that $H^1(X, \mathcal{O}_X) = 0$ (Prop. 3.18), we obtain from Th. 1.3 the exact sequence

$$0 \to H^0(X, L^\vee) \to H^0(X, \mathcal{O}_X) \to H^0(X, \iota_*(\mathcal{O}_{Z(\sigma)})) = H^0(Z(\sigma), \mathcal{O}_{Z(\sigma)}) = V_\sigma \to H^1(X, L^\vee) \to 0$$

and thus since $\dim_{\mathbb{C}}(H^0(X, \mathcal{O}_X)) = 1$, we see that

$$\dim_{\mathbb{C}}(V_\sigma) = 1 - \dim_{\mathbb{C}}(H^0(X, L^\vee)) + \dim_{\mathbb{C}}(H^1(X, L^\vee))$$

which shows that $\dim_{\mathbb{C}}(V_\sigma)$ is independent of σ.

Exercise 1.4. *Let L, M be line bundles on $\mathbb{P}^1_{\mathbb{C}}$. Suppose that $\Gamma(\mathbb{P}^1_{\mathbb{C}}, L) \neq 0$ and that $\Gamma(\mathbb{P}^1_{\mathbb{C}}, M) \neq 0$. Suppose that L and M have the same degree. Prove that L is isomorphic to M as an $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$-module.*

Let J be a line bundle on $X = \mathbb{P}^1_{\mathbb{C}}$. We shall consider the Cech complex associated with J. Let

$$U_0 = \operatorname{Spec} \mathbb{C}[X/Y],\ U_1 = \operatorname{Spec} \mathbb{C}[Y/X]$$

be the standard covering of X by affine open subschemes. Recall (see proof of Th. 3.4) that U_0 and U_1 are glued along the affine open subscheme $\operatorname{Spec} \mathbb{C}[X/Y]_{X/Y} = U_0 \cap U_1$, which is an open subscheme of U_0 (resp. U_1) via the natural inclusion of rings $\mathbb{C}[X/Y] \to \mathbb{C}[X/Y]_{X/Y}$ (resp. $\mathbb{C}[Y/X] \to \mathbb{C}[X/Y]_{X/Y}$). Since $\mathbb{C}[X/Y]$ and $\mathbb{C}[Y/X]$ are PIDs, the structure theorem for finitely generated modules over PIDs shows that the restrictions of J to U_0 and U_1 are trivial line bundles. Choose identifications $J|_{U_0} \simeq \mathcal{O}_{U_0}$ and $J|_{U_1} \simeq \mathcal{O}_{U_1}$. The structure of line bundle of J on X is now entirely given by the glueing map $(J|_{U_1})|_{U_0 \cap U_1} \to (J|_{U_0})|_{U_0 \cap U_1}$ which under the above conventions corresponds to a unit R in $\mathbb{C}[X/Y]_{X/Y}$. Note that multiplying R by a non zero complex number $\mu \in \mathbb{C}$ gives an isomorphic line bundle, because if we replace the identification $J|_{U_1} \simeq \mathcal{O}_{U_1}$ by the the same identification multiplied by μ then this will lead to the unit $\mu \cdot R$.

If we now apply the global sections functor $H^0(X, \bullet)$ to the corresponding Cech complex we obtain a sequence of the form

$$\cdots \to 0 \to \mathbb{C}[X/Y] \oplus \mathbb{C}[Y/X] \overset{P \oplus Q \mapsto P - R \cdot Q}{\to} \mathbb{C}[X/Y]_{X/Y} \to 0 \to \dots$$

Writing $z = X/Y$ for ease notation, this can be rewritten

$$\dots 0 \to 0 \to \mathbb{C}[z] \oplus \mathbb{C}[1/z] \overset{P \oplus Q \mapsto P - R \cdot Q}{\to} \mathbb{C}[z]_z \to 0 \to 0 \to \dots \tag{1}$$

with R a unit in $\mathbb{C}[z]_z$. As usual we adopt the convention that this complex is of index 0 at the term $\mathbb{C}[z] \oplus \mathbb{C}[1/z]$.

Note that $\mathbb{C}[z]_z$ is the subring of the fraction field of $\mathbb{C}[z]$ corresponding to the Laurent power series in z with finitely positive as well as negative elements (also called "Laurent polynomials").

We now record the

Lemma 1.1. *The units in $\mathbb{C}[z]_z$ are the elements of the form $\lambda \cdot z^k$ where $k \in \mathbb{Z}$ and $\lambda \in \mathbb{C}\backslash\{0\}$.*

Proof. (of the lemma). Let $A, B \in \mathbb{C}[z]_z$ be such that $A \cdot B = 1$. We may suppose without restriction of generality that A is monic. Let $i \geq 0$ be such that $z^i \cdot A \in \mathbb{C}[z]$ and $z^i \cdot A \in \mathbb{C}[z]$. Then we have $(z^i \cdot A) \cdot (z^i \cdot B) = z^{2i}$. By unique factorisation, $z^i \cdot A$ is thus a positive power of z and thus A is an integer power of z. QED

From the lemma, we deduce that $R = \lambda \cdot z^k$ for some $k \in \mathbb{Z}$ and $\lambda \in \mathbb{C}\backslash 0$. We now compute the homology of the complex (1).

The homology at the index 0 is the kernel of the map $P \oplus Q \mapsto P - R \cdot Q$. This by definition consists of the pairs $(P \in \mathbb{C}[z], Q \in \mathbb{C}[1/z])$, such that $P = \lambda \cdot z^k \cdot Q$, or in other words of the pairs $(P \in \mathbb{C}[z], \lambda^{-1} z^{-k} P)$ where $P \in \mathbb{C}[z]$ is such that $\lambda^{-1} z^{-k} P \in \mathbb{C}[1/z]$. By inspection, this corresponds to the polynomials $P \in \mathbb{C}[z]$ of degree $\leq k$ (with the convention that the 0 polynomial 0 has degree $-\infty$). This space has dimension $k+1$ if $k \geq 0$ and dimension 0 otherwise.

The homology at the index 1 is the quotient of $\mathbb{C}[z]_z$ by the image of the map $P \oplus Q \mapsto P - R \cdot Q$. Since the image of this map contains $\mathbb{C}[z]$, we may identify this quotient with the quotient of the vector space $\mathbb{C}[z]_{\leq 0}$ of Laurent polynomials with only negative terms by the vector space $R \cdot \mathbb{C}[z] \cap \mathbb{C}[z]_{\leq 0}$. Now $R \cdot \mathbb{C}[z] \cap \mathbb{C}[z]_{\leq 0}$ consists precisely of the Laurent polynomials of the form $T(1/z)$, where $T \in \mathbb{C}[z]$ is a polynomial of degree at most $-k$ (with again the convention that the 0 polynomial 0 has degree $-\infty$). This space has dimension $-k-1$ if $k \leq 0$ and has dimension 0 otherwise.

We conclude that

$$\dim_{\mathbb{C}}(H^0(X, J)) - \dim_{\mathbb{C}}(H^1(X, J)) = k + 1 \tag{2}$$

Coming back to the original problem, suppose that L and M have the same degree. Then from the solution of Ex. 1.3 of this sheet, we see that

$$\dim_{\mathbb{C}}(H^0(X, L^\vee)) - \dim_{\mathbb{C}}(H^1(X, L^\vee)) = \dim_{\mathbb{C}}(H^0(X, M^\vee)) - \dim_{\mathbb{C}}(H^1(X, M^\vee)).$$

From equation (2) we see that the unit associated with the glueing map of $L^\vee$ is of the form

$$\lambda \cdot z^{\dim_{\mathbb{C}}(H^0(X, L^\vee)) - \dim_{\mathbb{C}}(H^1(X, L^\vee)) - 1}$$

and by the above remark we may assume without restriction of generality that $\lambda = 1$. The same applies to $M^\vee$. Thus the glueing data of $L^\vee$ and $M^\vee$ are the same and they are thus isomorphic. Since $(L^\vee)^\vee \simeq L$ and $(M^\vee)^\vee \simeq M$ we conclude that L and M are isomorphic.