Solution of exercise 1.2 of sheet 2*

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Lemma 1.1. *Let Z be a scheme and let (V_j) be a finite open covering of Z. Suppose that all the V_j are quasi-compact. Then Z is quasi-compact. In particular, Z is quasi-compact if and only if it has a finite open affine cover.*

Proof. Let (O_j) be an open covering of Z. For each k, there is a finite subfamily $O_{j_1}, \dots, O_{j_l}$ such that the open sets $(O_{j_r} \cap V_k)_{r \in \{1,\dots,l\}}$ covers V_k. Since (V_k) is a finite family, the family $O_{j_1}, \dots, O_{j_l}$ may be taken independent of k. The family $O_{j_1}, \dots, O_{j_l}$ then covers Z. Since (O_j) was arbitrary, we have shown that Z is quasi-compact.

Thus, since affine schemes are quasi-compact, we have shown that if Z has a finite open affine cover, then Z is quasi-compact. Finally, if Z is quasi-compact, then it clearly has a finite open affine cover. □

Lemma 1.2. *Let $f : X \to Y$ be a morphism of schemes. Suppose that there is (U_i) an open affine covering of Y such that $f^{-1}(U_i)$ is quasi-compact. Then for any open affine subscheme $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact.*

Proof. Suppose first that Y is affine. There is then a finite open affine subcovering of (U_i), since Y is quasi-compact, so we may suppose wrog that (U_i) is finite. Let (V_{ik}) be a finite open affine covering of $f^{-1}(U_i)$ for each i. This exists by assumption. The finite family of the $V_{i_k} \cap f^{-1}(U) \simeq V_{i_k} \times_Y U$ is then a finite covering of $f^{-1}(U)$. Furthermore $V_{i_k} \times_Y U$ is affine because V_{i_k}, Y and U are all affine (look at the construction of the fibre product). Hence, by Lemma 1.1, we have proven the lemma when Y is affine.

In the general case let (S_{ik}) be an open affine covering of $U \cap U_i$ for each i. By the special case proven in the last paragraph, $f^{-1}(S_{ik})$ is quasi-compact, since S_{ik} is an open affine subscheme of U_i, U_i is affine and $f^{-1}(U_i)$ is quasi-compact. Thus, since the S_{ik} cover U and U is affine, we deduce from the same special case that $f^{-1}(U)$ is quasi-compact. □

Lemma 1.3. *Let $f : X \to Y$ be a morphism of schemes. Suppose that there is (U_i) an open affine covering of Y such that $f^{-1}(U_i)$ is affine. Then for any open affine subscheme $U \subseteq Y$, the open subscheme $f^{-1}(U)$ is affine.*

Proof. Suppose first that Y is affine. For each U_i, let $(D_{f_{ik}})$ be a covering of $U_i \cap U$ by basic open sets of Y. Then $f^{-1}(U)$ is covered by sets of the form $f^{-1}(D_{f_{ik}})$. Furthermore, we have

$$f^{-1}(D_{f_{ik}}) = f^{-1}(U_i) \times_Y D_{f_{ik}} = f^{-1}(U)_{f^*(f_{ik})}$$

and thus in particular $f^{-1}(D_{f_{ik}})$ is affine. Notice also that by assumption the $f_{ik}|_U$ generate the unit ideal of $\Gamma(U, \mathcal{O}_U)$. Hence the functions $f^*(f_{ik})|_{f^{-1}(U)}$ generate the unit ideal of $\Gamma(f^{-1}(U), \mathcal{O}_X)$. Furthermore,

*For the course C2.6 Introduction to Schemes, Oxford, Hilary Term 2017 a finite subfamily of the $f^{-1}(D_{f_{ik}})$ covers $f^{-1}(U)$ (since a finite subfamily of the $D_{f_{ik}}$ covers U, as U is quasi-compact) and for any two pairs of indices i_1k_1 and i_1k_2, we have

$$f^{-1}(D_{f_{i_1k_1}}) \cap f^{-1}(D_{f_{i_2k_2}}) = f^{-1}(D_{f_{i_1k_1}}) \times_Y f^{-1}(D_{f_{i_2k_2}})$$

which is affine. We conclude that $f^{-1}(U)$ is affine by appealing to Corollary 2.39 (and complement 2.38).

In the general case, let as before (S_{ik}) be an open affine covering of $U \cap U_i$ for each i. By the special case proven in the last paragraph, $f^{-1}(S_{ik})$ is affine, since it is an open affine subscheme of U_i, U_i is affine and $f^{-1}(U_i)$ is affine. Thus by the same special case, $f^{-1}(U)$ is affine. □

Lemma 1.4. *Let $f : X \to Y$ be a morphism of schemes. Suppose that there is an open affine covering (U_i) of Y and an open affine covering (V_{ik}) of X such that $f(V_{ik}) \subseteq U_i$ and $\mathcal{O}_X(V_{ik})$ is a finitely generated $\mathcal{O}_Y(U_i)$-algebra.*

Then for any open affine subscheme $U \subseteq Y$, there is an open affine covering S_l of $f^{-1}(U)$ such that $\mathcal{O}_X(S_l)$ is a finitely generated $\mathcal{O}_Y(U)$-algebra.

Proof. Let $D_{f_{ij}}$ be an open affine covering of $U_i \cap U$ by basic open sets of U. The schemes $D_{f_{ij}} \times_{U_i} V_{ik}$ are then affine and

$$\mathcal{O}_X(D_{f_{ij}} \times_{U_i} V_{ik}) = \mathcal{O}_Y(D_{f_{ij}}) \times_{\mathcal{O}_Y(U_i)} \mathcal{O}_X(V_{ik})$$

is then a finitely generated $\mathcal{O}_Y(D_{f_{ij}})$-algebra. Now $\mathcal{O}_Y(D_{f_{ij}})$ is a finitely generated $\mathcal{O}_Y(U)$-algebra and thus $\mathcal{O}_X(D_{f_{ij}} \times_{U_i} V_{ik})$ is a finitely generated $\mathcal{O}_Y(U)$-algebra. Since the $D_{f_{ij}}$ cover U, this proves the lemma. □