Noncommutative Rings

Solutions to Problem Sheet 4

1. Let $A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and let $P = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Show that P is a prime ideal in A. Also, show that $S := A \backslash P$ is multiplicatively closed but is not a right Ore set. Prove that S *is* a left localisable subset of A and that $S^{-1}A \cong \mathbb{Q}$.

   Solution. The map $\theta : A \to \mathbb{Z}$ which sends $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ to c is a surjective ring homomorphism with kernel P. Since $\mathbb{Z}$ is an integral domain, we deduce that $xy \in P$ forces $x \in P$ or $y \in P$ (such ideals in the non-commutative world are called *completely prime*). Now if I, J are ideals in A such that $IJ \subseteq P$ then for all $x \in P$, $xJ \subseteq P$ forces either $x \in P$ or $J \subseteq P$. Hence $I \subseteq P$ or $J \subseteq P$ and P is prime.

   Note that $S = \theta^{-1}(\mathbb{Z}\backslash\{0\})$. If $\theta(x) \neq 0$ and $\theta(y) \neq 0$ then $\theta(xy) = \theta(x)\theta(y) \neq 0$. So $x, y \in S$ implies $xy \in S$; also $1 \in S$ so S is indeed multiplicatively closed. To see that S is not a right Ore set, consider the elements

$$s := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in S \quad \text{and} \quad a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in A.$$

   Then for any $s' := \begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix} \in S$ and $a' := \begin{pmatrix} u' & v' \\ 0 & w' \end{pmatrix} \in A$ we have

$$sa' = \begin{pmatrix} 0 & 0 \\ 0 & w' \end{pmatrix} \quad \text{and} \quad as' = \begin{pmatrix} 0 & z' \\ 0 & 0 \end{pmatrix}.$$

   Since $z' \neq 0$ because $s' \in S$, we see that sa' cannot be equal to as'. Thus S is not a right Ore set.

   To show that S is a left localisable, we could note that A is a finitely generated $\mathbb{Z}$ module, hence a left Noetherian ring, so by Proposition 3.11 and Theorem 3.8 it is enough to show that S is a left Ore set. This can be done by a direct verification. Here is an alternative, perhaps a little more conceptual, solution, which also has the advantage of simultaneously computing $S^{-1}A$.

   Let $\varphi : A \to S^{-1}A$ and $\iota : \mathbb{Z} \to \mathbb{Q}$ be the localisation maps. Since $\iota \circ \theta : A \to \mathbb{Q}$ sends S to units, by the universal property of $S^{-1}A$ there is some $\psi : S^{-1}A \to \mathbb{Q}$ such that $\psi \circ \varphi = \iota \circ \theta$:

$$\begin{pmatrix} A[r]^{\varphi}[d]_{\theta} & S^{-1}A[d]^{\psi} \\ \mathbb{Z}[r]_{\iota} & \mathbb{Q} \end{pmatrix}$$

   Now if $\varphi(a) = 0$ then $\iota\theta(a) = \psi(\varphi(a)) = 0$ so $a \in \ker\theta = P$. On the other hand,

$$sP = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\varphi(P) = \varphi(s)^{-1}\varphi(sP) = 0$ and $P = \ker\varphi$. Since $A/P \cong \operatorname{im}\theta = \mathbb{Z}$, there is a map $\sigma : \mathbb{Z} \to S^{-1}A$ such that $\varphi = \sigma \circ \theta$:

$$\begin{pmatrix} A[r]^{\varphi}[d]_{\theta} & S^{-1}A[d]^{\psi} \\ \mathbb{Z}[ur]_{\sigma}[r]_{\iota} & \mathbb{Q} \end{pmatrix}.$$

If $n \in \mathbb{Z}$ then $\sigma(n) = \sigma(\theta(\tilde{n})) = \varphi(\tilde{n})$ where $\tilde{n} := \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}$. Since $\widetilde{n} \in S$ whenever $n \neq 0$, by the universal property of $\mathbb{Q}$ there is a map $\widetilde{\sigma} : \mathbb{Q} \to S^{-1}A$ such that $\sigma = \widetilde{\sigma} \circ \iota$. Now

$$\widetilde{\sigma}\psi\varphi = \widetilde{\sigma}\iota\theta = \sigma\theta = \varphi$$

so the ring endomorphism $\widetilde{\sigma}\psi : S^{-1}A \to S^{-1}A$ is the identity map on $\varphi(A)$. So it is also the identity map on $\varphi(S)^{-1}$. But $S^{-1}A$ is generated as a ring by these two sets, so $\widetilde{\sigma}\psi$ is the identity map on all of $S^{-1}A$. It follows that ψ is injective. Now

$$n^{-1}m = \psi(\varphi(\widetilde{n})^{-1}\varphi(\widetilde{m}))$$

for any $n^{-1}m \in \mathbb{Q}$ so ψ is surjective. Hence $\psi : S^{-1}A \to \mathbb{Q}$ is an isomorphism. It follows that every element of $S^{-1}A$ can be written in the form $\varphi(\widetilde{n})^{-1}\varphi(\widetilde{m})$ for some $n, m \in \mathbb{Z}$ with $n \neq 0$. Finally, we saw above that $\ker\varphi = P$, which is left S-torsion because $sP = 0$. So $\ker\varphi = t_S(A)$ and S is left localisable, directly from Definition 3.4(a).

2. Suppose that A is left Noetherian, and let S be a left localisable subset of A.

   (a) Prove that $Q := S^{-1}A$ is also left Noetherian.

   (b) Show that if I is a two-sided ideal in A then $Q \cdot I$ is also a two-sided ideal in Q.

   (c) Suppose further that A is also right Noetherian, and that P is a prime ideal in A such that $P \cap S = \emptyset$. Show that $Q \cdot P$ is a prime ideal in Q.

Solution. (a) Let I be a left ideal in Q, and let $\varphi : A \to Q = S^{-1}A$ be the localisation map. Then $J := \varphi^{-1}(I)$ is a left ideal in A. Since A is left Noetherian, J is generated by some finite subset X of A. Clearly $Q \cdot \varphi(X) \subseteq I$. Now if $x \in I$ then $\varphi(s)x \in \varphi(A)$ for some $s \in S$, so $\varphi(s)x = \varphi(u)$ for some $u \in J$. This shows that $x = \varphi(s)^{-1}\varphi(u) \in Q \cdot \varphi(J) = Q \cdot \varphi(A) \cdot \varphi(X) = Q \cdot \varphi(X)$. Thus $I = Q \cdot \varphi(X)$ is finitely generated, so Q is left Noetherian.

(b) We have to show that $Q \cdot I = Q\varphi(I)$ is a *right* ideal in Q. Since it is clearly stable under right multiplication by $\varphi(A)$, we have to show that $QI \cdot s^{-1} \subseteq QI$ for all $s \in \varphi(S)$. Consider the ascending chain of left ideals $QI \leqslant QIs^{-1} \leqslant QIs^{-2} \leqslant \cdots$. This chain terminates because Q is left Noetherian by part (a). So $QIs^{-n} = QIs^{-n-1}$ for some $n \in \mathbb{N}$. Multiplying this on the right by s^n shows that $QIs^{-1} = QI$, as required.

(c) By part (b), $Q \cdot P$ is a two-sided ideal in Q. Suppose $Q \cdot P = Q$. Then $1 \in Q\varphi(P)$ so $\varphi(s) \in \varphi(P)$ for some $s \in S$. Hence $\varphi(s - u) = 0$ for some $u \in P$ so $t(s - u) = 0$ for some $t \in S$. But then $ts = tu \in S \cap P$, a contradiction. Thus $Q \cdot P$ is a *proper* ideal of Q.

Consider the two-sided ideal $K := \varphi^{-1}(QP)$ of A. If $u \in K$ then $\varphi(u) \in QP$ so $\varphi(su) \in P$ for some $s \in S$, so $tsu \in P$ for some $t \in S$. Thus, for all $u \in K$ there exists $x \in S$ such that $xu \in P$. Since A is right Noetherian and K is a right ideal, we can write $K = u_1A + \cdots + u_mA$ for some $u_1, \ldots, u_m \in K$. Using the left Ore condition, we can find $s \in S$ such that $su_i \in P$ *for all* $i = 1, \ldots, m$. Therefore $sK = su_1A + \cdots + su_mA \subseteq P$. But P is prime, so $(RsR) \cdot K \subseteq P$ forces $s \in P$ or $K \subseteq P$. Since $S \cap P = \emptyset$ by assumption, we deduce that $K = P$. Now suppose that $I, J \lhd Q$ are such that $IJ \subseteq QP$. Then $\varphi^{-1}(I)\varphi^{-1}(J) \subseteq \varphi^{-1}(QP) = K = P$ so either $\varphi^{-1}(I) \subseteq P$ or $\varphi^{-1}(J) \subseteq P$. But $I = Q \cdot \varphi(\varphi^{-1}(I))$ and $J = Q \cdot \varphi(\varphi^{-1}(J))$ by part (a), so we deduce that either $I \subseteq QP$ or $J \subseteq QP$.

3. Let A be a filtered ring and let M be a filtered left A-module.

   (a) Show that $\widetilde{M}/t\widetilde{M}$ is isomorphic to $\operatorname{gr} M$ as a left $\operatorname{gr} A$-module.

   (b) Viewing M as a left $\widetilde{A}$-module via the isomorphism $\widetilde{A}/(t-1)\widetilde{A} \cong A$ from Lemma 4.20(2), show that $\widetilde{M}/(t-1)\widetilde{M}$ is isomorphic to M as a left $\widetilde{A}$-module.

Solution. (a) Define $\pi : \widetilde{M} \to \operatorname{gr} M$ on homogeneous elements by $\pi(m_it^i) = m_i + M_{i-1}$, and extend to the whole of $\widetilde{M}$. It is a surjective map on homogeneous components, hence surjective. If $\sum m_it^i \in \ker \pi$ then $\sum m_i + M_{i-1} = 0$ in $\operatorname{gr} M$, so that $m_i \in M_{i-1}$ for all i. But then $\sum m_it^i = t \cdot \sum m_it^{i-1} \in t\widetilde{M}$. So $\ker \pi = t\widetilde{M}$. Hence $\overline{\pi} : \widetilde{M}/t\widetilde{M} \longrightarrow \operatorname{gr} M$ given on homogeneous elements by $\overline{\pi}(m_it^i + t\widetilde{M}) = m_i + M_{i-1}$ is an isomorphism of abelian groups. It remains to check that this isomorphism is compatible with the left $\widetilde{A}/t\widetilde{A} \cong \operatorname{gr} A$-module structures:

$$\overline{\pi}((a_it^i + t\widetilde{A}) \cdot (m_jt^j + t\widetilde{M})) = \overline{\pi}(a_im_jt^{i+j} + t\widetilde{M}) = a_im_j + M_{i+j-1}$$

whereas

$$\overline{\pi}(a_it^i + t\widetilde{A}) \cdot \overline{\pi}(m_jt^j + t\widetilde{M}) = (a_i + A_{i-1}) \cdot (m_j + M_{j-1}) = a_im_j + M_{i+j-1}.$$

(b) The map $\theta_M : M[t, t^{-1}] \to M$ which is the identity on M and which sends t^i to 1 for all $i \in \mathbb{Z}$ has kernel $(t-1)M[t, t^{-1}]$. A similar argument to the one given in the proof of Lemma 4.20(2) also shows that $(t-1)M[t, t^{-1}] \cap \widetilde{M} = (t-1)\widetilde{M}$. Since θ_M is onto, we see that the map $\overline{\theta_M} : \widetilde{M}/(t-1)\widetilde{M} \to M$ which sends $\sum m_it^i + (t-1)\widetilde{M}$ to $\sum m_i$ is an isomorphism of abelian groups. It remains to check that it is compatible with the isomorphism $\overline{\theta} : \widetilde{A}/(t-1)\widetilde{A} \to A$:

$$\overline{\theta_M}((a_it^i + (t-1)\widetilde{A})(m_jt^j + (t-1)\widetilde{M})) = a_im_j$$

whenever $a_i \in A_i$ and $m_j \in M_j$, and

$$\overline{\theta}(a_it^i + (t-1)\widetilde{A}) \cdot \overline{\theta_M}(m_jt^j + (t-1)\widetilde{M}) = a_i \cdot m_j = a_im_j.$$

Hence $\widetilde{M}/(t-1)\widetilde{M}$ is isomorphic to M as a left $\widetilde{A}$-module.

4. (a) Verify that the commutator bracket on a ring A is a Poisson bracket.

   (b) Let k be a field. Suppose that $\{,\}$ is a Poisson bracket on the polynomial ring $A = k[x_1, \ldots, x_n]$ such that $\{k, A\} = 0$. Prove that $\{,\}$ is completely determined by its values on the x_i's.

   (c) Let A be a filtered ring such that $\operatorname{gr} A$ is commutative, and let $\{,\}$ be the induced Poisson bracket on $\operatorname{gr} A$. Show that $\operatorname{gr} I$ is closed under $\{,\}$ for any left ideal I in A.

(d) Find an example of a filtered ring A and a graded ideal J in $\operatorname{gr} A$ such that $\operatorname{gr} A$ is commutative and $\{J, J\} \subseteq J$ but $\{\sqrt{J}, \sqrt{J}\} \not\subseteq \sqrt{J}$.

Solution. (a) This is straightforward: expand everything out and compare sides. For example:

$$[x, yz]-y[x, z]-[x, y]z = x(yz)-(yz)x-y(xz-zx)-(xy-yx)z = xyz-yzx-yxz+yzx-xyz+yxz = 0.$$

(b) Since $\{k, A\} = 0$ we see that $\{\lambda x, y\} = \{\lambda, y\}x + \lambda\{x, y\} = \lambda\{x, y\}$ for all $x, y \in A$ and $\lambda \in k$. Thus $\{,\}$ is k-bilinear because it is anti-symmetric by definition. Since A is spanned by the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ as a k-vector space, the Poisson bracket is completely determined by its values on such monomials. Now

$$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n}, a\} = \sum_{i=1}^{n} \alpha_i x_1^{\alpha_1} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n} \{x_i, a\}$$

shows that the Poisson bracket is completely determined by terms of the form $\{x_i, x_1^{\beta_1} \cdots x_n^{\beta_n}\}$. But this is equal to

$$\sum_{j=1}^{n} \{x_i, x_j\} \beta_j x_1^{\beta_1} \cdots x_j^{\beta_j - 1} \cdots x_n^{\beta_n}$$

so we see that it's enough to know $\{x_i, x_j\}$ to calculate all possible values of $\{,\}$.

(c) Let X, Y be homogeneous elements of $\operatorname{gr} I$ of degrees i, j respectively. Then $X \in ((I \cap R_i) + R_{i-1})/R_{i-1}$, so we can write $X = x + R_{i-1}$ for some $x \in I \cap R_i$ and similarly $Y = y + R_{j-1}$ for some $y \in I \cap R_j$. Now on the one hand, $[x, y] = x \cdot y + (-y) \cdot x \in I$ because I is a left ideal, and on the other hand, $[x, y] \in R_{i+j-1}$ because $\operatorname{gr} R$ is commutative. Thus $\{X, Y\} = [x, y] + R_{i+j-2} \in ((I \cap R_{i+j-1}) + R_{i+j-2})/R_{i+j-2} \subset \operatorname{gr} I$. So $\{\operatorname{gr} I, \operatorname{gr} I\} \subseteq \operatorname{gr} I$.

(d) Let $A = A_1(k)$ be the first Weyl algebra so that $\operatorname{gr} A = k[X, Y]$ is a commutative polynomial ring in two variables, and let $J = \mathfrak{m}^2$ where $\mathfrak{m}$ is any ideal of $\operatorname{gr} A$. Then $\{J, J\} = \{\mathfrak{m}^2, J\} \subseteq \mathfrak{m}\{\mathfrak{m}, J\} + \{\mathfrak{m}, J\}\mathfrak{m}$ and $\{\mathfrak{m}, J\} = \{\mathfrak{m}, \mathfrak{m}^2\} = \mathfrak{m}\{\mathfrak{m}, \mathfrak{m}\} + \{\mathfrak{m}, \mathfrak{m}\}\mathfrak{m} \subseteq \mathfrak{m}$. Hence $\{J, J\} \subseteq J$ regardless of what $\mathfrak{m}$ is. Now take $\mathfrak{m} = \langle x, y \rangle \lhd A$. Then $J := \mathfrak{m}^2 = \langle x^2, xy, y^2 \rangle$ is a graded ideal of $\operatorname{gr} A$ closed under the Poisson bracket, but $\sqrt{J} = \mathfrak{m}$ is *not* closed under the Poisson bracket because $\{y, x\} = 1 \notin \mathfrak{m}$.

5. Let B be a left Noetherian ring, and let $t \in B$ be a central regular element. By considering the ring $(t^{\mathbb{N}})^{-1}B$ or otherwise, show that for any left ideal I of B there is an integer n such that $I \cap t^n B \subseteq tI$.

   Solution. Since t is regular, we can view B as a subring of $B_t = (t^{\mathbb{N}})^{-1}B$. Consider the chain $I \subseteq t^{-1}I \cap B \subseteq t^{-2}I \cap B \subseteq \cdots$ of left ideals of B. Since B is left Noetherian, for some $n \geq 1$ we have that $t^{-(n-1)}I \cap B = t^{-n}I \cap B$. Multiplying this relation by t^n shows that $I \cap t^n B \subseteq t^n(t^{-(n-1)}I) = tI$.

6. Let $n \geq 1$, and let k be a field of characteristic zero. Show that there are no $n \times n$ matrices X, Y with entries in k that satisfy the relation $YX - XY = 1$. What happens if the characteristic of k is positive?

   Solution. If $YX - XY = 1$ then taking traces shows that $\operatorname{tr}(YX - XY) = \operatorname{tr}(1) = n$. But $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ so $n = 0$, a contradiction. If the characteristic of k is positive, then it is possible to find such matrices: the problem is equivalent to finding a non-zero finite dimensional module over the first Weyl algebra $A := A_1(k)$. Now by Exercise 1.1, inside A we have

$$y^p x = \sum_{i=0}^{p} \binom{p}{i} \operatorname{ad}_y^i(x) y^{p-i} = xy^p + \operatorname{ad}_y^p(x) = xy^p$$

because $\operatorname{ad}_y^2(x) = [y,[y,x]] = [y,1] = 0$. Hence y^p is central in $A_1(k)$ and a similar argument shows that x^p is central. Let $V = A/(Ax^p + Ay^p)$. It follows from Exercise 1.3(b) that the image of $\{x^i y^j : 0 \leqslant i,j \leqslant p-1\}$ in V is a k-vector space basis for V. So V is a p^2-dimensional k-vector space and also a left A-module. If $X, Y \in \operatorname{End}_k(V) \cong M_{p^2}(k)$ are the matrices that give the action of $x, y \in A$ on V respectively, then $YX - XY = 1$ inside $M_{p^2}(k)$.

7. Let R be a filtered ring, let M be a filtered left $R-$module with filtration $(M_i)_{i\in\mathbb{Z}}$ and let N be a submodule of M. Equip N with the *subspace filtration* $N_i := N \cap M_i$, and equip M/N with the *quotient filtration* $(M/N)_i := (M_i + N)/N$. Show that

   (a) there is an injective $\operatorname{gr} R$-module homomorphism $\alpha : \operatorname{gr} N \to \operatorname{gr} M$,

   (b) there is a surjective $\operatorname{gr} R$-module homomorphism $\beta : \operatorname{gr} M \to \operatorname{gr}(M/N)$,

   (c) $\ker \beta = \operatorname{Im} \alpha$.

   Solution: (a) The natural composition of maps $N_i \hookrightarrow M_i$ and $M_i \twoheadrightarrow M_i/M_{i-1}$ has kernel $N_i \cap M_{i-1} = N \cap M_{i-1} = N_{i-1}$. So we have an injection of abelian groups

   $$\alpha_i : N_i/N_{i-1} \hookrightarrow M_i/M_{i-1}$$

   for all $i \in \mathbb{Z}$. Putting these together we get an injection

   $$\alpha = \oplus \alpha_i : \operatorname{gr} N \to \operatorname{gr} M.$$

   You should now check that α is a left $\operatorname{gr} R-$module homomorphism.

   (b) Let β_i be the composition

   $$\beta_i : M_i/M_{i-1} \xrightarrow{u_i} \frac{M_i + N}{M_{i-1} + N} \xrightarrow{v_i} \frac{(M_i + N)/N}{(M_{i-1} + N)/N}$$

   where $u_i(m + M_{i-1}) = m + M_{i-1} + N$ and v_i is the natural isomorphism. Then β_i is surjective, hence so is

   $$\beta := \oplus \beta_i : \operatorname{gr} M \to \operatorname{gr}(M/N).$$

   You should now check that β is a left $\operatorname{gr} R$-module homomorphism.

   (c) Recall the *modular law*, which states that if X, Y, Z are three subgroups of some larger abelian group with $X \subseteq Y$, then

   $$Y \cap (X + Z) = X + (Y \cap Z).$$

   Using this fact, we see that

   $$\ker(u_i) = \frac{M_i \cap (M_{i-1} + N)}{M_{i-1}} = \frac{M_{i-1} + (M_i \cap N)}{M_{i-1}} = \frac{M_{i-1} + N_i}{M_{i-1}} = \operatorname{im}(\alpha_i)$$

   Since v_i is an isomorphism, it follows that $\ker(\beta_i) = \operatorname{im}(\alpha_i)$ for all $i \in \mathbb{Z}$. Hence $\ker(\beta) = \operatorname{im}(\alpha)$.

8. Let $A = A_n(k)$ be the Weyl algebra, and let r be an integer such that $n \leqslant r \leqslant 2n$. Give an example of a cyclic A-module M such that $d(M) = r$. Justify your answer.

Solution. Fix $d = 0, \ldots, n$, let $I = Ay_1 + \cdots + Ay_d$, and let $J := (\operatorname{gr} A)Y_1 + \cdots (\operatorname{gr} A)Y_d$. We will show that $\operatorname{gr} I = J$, the $\supseteq$ inclusion being clear. The monomials $\{y^\beta : \beta \in \mathbb{N}^n\}$ form a basis for A as a left $R = k[x_1, \ldots, x_n]$-module by Exercise 1.3(b). Hence $\{y^\beta : y_i \geq 1 \text{ for some } 1 \leqslant i \leqslant d\}$ forms a basis for I as a left R-module. Let $u \in I$ have degree m; then we can write $u = \sum_{|\beta|=m} u_\beta y^\beta$ for some $u_\beta \in R$ such that $u_\beta = 0$ whenever $\beta_1 = \cdots = \beta_d = 0$. Hence $u + A_{m-1} = \sum_{|\beta|=m} u_\beta \cdot Y^\beta \in J$, because $u_\beta \neq 0$ forces $Y_i | Y^\beta$ for some $1 \leqslant i \leqslant d$ and then $Y^\beta \in J$. Thus $\operatorname{gr} I \subseteq J$ as claimed. The ideal J is prime, and the ring $\operatorname{gr} A/J$ is isomorphic to a polynomial ring over k in $n+(n-d) = 2n-d$ variables. Hence $\operatorname{Kdim}(\operatorname{gr} A/J) = 2n - d$.

Let $d := 2n - r$. Since $n \leqslant r \leqslant 2n$, we have $0 \leqslant d \leqslant n$. Let $M := A/I$; then $\operatorname{gr} M \cong \operatorname{gr} A/ \operatorname{gr} I$ by Question 5, so $d(M) = \operatorname{Kdim}(\operatorname{gr} A/ \operatorname{gr} I) = 2n - (2n - r) = r$.