C2.5 Non-commutative rings

Problem Sheet 0

All rings are assumed to be associative and containing 1.

1. Let R be a simple ring (that is, the only 2-sided ideals of R are $\{0\}$ and R). Show that the centre of R is a field.

2. Give an example of a ring R with elements $a, b \in R$ such that $ab = 1$ but $ba \neq 1$.

3. Let R be a ring which satisfies one of the conditions (a) or (b) below.

   (a) The descending chain condition on left ideals (that is, any descending chain of left ideals of R stabilizes), or

   (b) The ascending chain condition on left ideals.

   Show that if $a, b \in R$ are such that $ab = 1$ then $ba = 1$.

4. Suppose that the elements x, y of a ring R are such that $1 - xy$ has a right inverse (that is, an element $z \in R$ such that $(1 - xy)z = 1$). Show that $1 - yx$ has a right inverse.

5. Now suppose that R is a commutative ring which is generated by n elements as an algebra over a field F. Let M be a maximal ideal of R. Show that M can be generated by n elements as an ideal of R.