C2.3 Representations of semisimple Lie algebras

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Problem Sheet 4

1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

(i) Let L be a finite dimensional $\mathfrak{g}$-module. Show that L is simple if and only if the dual module L^* is simple.

(ii) Let $L(\lambda)$, $\lambda \in P^+$ be a simple $\mathfrak{g}$-module with highest weight λ. Show that the dual $L(\lambda)^*$ is isomorphic to $L(-w_0(\lambda))$, where w_0 is the Weyl group element sending Φ^+ (the positive roots) to $-\Phi^+$.

(iii) What condition should λ satisfy such that 0 is a weight of $L(\lambda)$?

Solution: (i) If L is a finite dimensional module, then $L^* = \{f : L \to \mathbb{C}\}$ as a vector space. The action on L^* is given by

$$(x \cdot f)(l) = -f(x \cdot l),\ x \in \mathfrak{g},\ l \in L,\ f \in L^*.$$

Define $A^\perp \subset L^*$ by $A^\perp = \{f \in L^* \mid f(a) = 0,\ \text{for all } a \in A\}$, for every submodule $A \subset L$. Then $A^\perp$ is a submodule of L^*, and if A is proper, then so is A^*. Hence if L is not simple, then L^* is not simple.

Conversely, if $B \subset L^*$ is a submodule, define $B^\perp \subset L$ by $B^\perp = \{l \in L \mid f(l) = 0,\ \text{for all } f \in B\}$ etc.

(ii) By (i), $L(\lambda)^*$ is a simple finite dimensional $\mathfrak{g}$-module. It is easy to see that the weights of $L(\lambda)^*$ are precisely the negatives of the weights of $L(\lambda)$. To see this, pick a basis of weight vectors in $L(\lambda)$ and take the dual basis in $L(\lambda)^*$; these are weight vectors of $L(\lambda)^*$. Moreover, we know that

$$\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)},\ \text{for all } w \in W,\ \mu \text{ a weight}.$$

This means that $\dim L(\lambda)_{w(\lambda)} = 1$ for all $w \in W$. In particular, $\dim L(\lambda)_{w_0(\lambda)} = 1$ and hence $\dim L(\lambda)^*_{-w_0(\lambda)} = 1$.

Notice that $-w_0(\lambda) \in P^+$ iff $\lambda \in P^+$. This is because $-w_0(\alpha) > 0$ for all $\alpha > 0$. Moreover, $-w_0(\lambda)$ is a maximal weight, otherwise there exists μ a weight of $L(\lambda)$ (so $-\mu$ is a weight of $L(\lambda)^*$) such that $-\mu-(-w_0(\lambda)) \in Q^+$. Apply $-w_0$ remembering that $-w_0(Q^+) = Q^+$ and it follows that $w_0(\mu)-\lambda \in Q^+$. But $w_0(\lambda)$ is also a weight of $L(\lambda)$, and this is a contradiction with the maximality of λ.

(iii) We know that if μ is a weight of $L(\lambda)$ then $\lambda - \mu \in Q^+$. This implies that a necessary condition for 0 to be a weight is that $\lambda \in Q^+$. In fact, this is also a sufficient condition, as we prove now.

To simplify the discussion, we call a subset S of P *saturated* (term due to Humphreys) if for every $\lambda \in S$ and every root $\alpha \in \Phi$ and every integer k between 0 and $\langle \lambda, \alpha \rangle$, the weight $\lambda - k\alpha$ is also in S. We claim that the set of weights $\Psi(L(\lambda))$ is saturated. To see this, fix a root α and regard $L(\lambda)$ as an sl_α module. By complete reducibility, this is a direct sum of simple finite dimensional sl_α modules and we know from the explicit description of these modules that the condition in the definition of saturated is satisfied.

Now, we prove that the fact that $\Psi(L(\lambda))$ is saturated implies that if $\mu \in P^+$ is such that $\mu < \lambda$, then μ is in $\Psi(L(\lambda))$. In particular, this works for $\mu = 0$ when $\lambda \in Q^+$.

Suppose μ' is a weight of $\Psi(L(\lambda))$ such that $\mu' > \mu$. ($\mu' = \lambda$ has this property and this is the starting point.) Then $\mu' = \mu + \sum_{\alpha \in \Pi} k_\alpha \alpha,\ k_\alpha \geq 0$. Suppose $\mu' \neq \mu$, then there exists some $k_\alpha > 0$. We want to find $\beta \in \Pi$ with $k_\beta > 0$ such that $\langle \mu', \beta \rangle > 0$. We have $\langle \sum k_\alpha \alpha, \sum k_\alpha \alpha \rangle > 0$ (since $\langle\ ,\ \rangle$ is positive definite), and therefore there must exist $\beta \in \Pi$ with $k_\beta > 0$ such that $\langle \sum k_\alpha \alpha, \beta \rangle > 0$. But then

$$\langle \mu', \beta \rangle = \langle \mu, \beta \rangle + \langle \sum k_\alpha \alpha, \beta \rangle > 0,$$

since $\langle \mu, \beta \rangle \geq 0$ as $\mu \in P^+$.

By the saturated property, $\mu' - \beta$ is also in $\Psi(L(\lambda))$ and moreover $\mu' - \beta \geq \mu$. In this way, we can reduce inductively the k_α's, until we find that $\mu \in \Psi(L(\lambda))$.

Notice that this also proves that a necessary and sufficient condition for $\lambda \in P$ to be a weight of $L(\lambda)$ is that $w(\mu) < \lambda$ for all $w \in W$.

□

2. Use the Weyl dimensional formula to show that for every natural number k, there exists a simple $\mathfrak{g}$-module of dimension k^r, where r is the number of positive roots of $\mathfrak{g}$.

Solution: Weyl's dimension formula says that

$$\dim L(\lambda) = \prod_{\alpha\in\Phi^+} \frac{\langle \lambda+\rho, \alpha\rangle}{\langle \rho, \alpha\rangle}.$$

Take $\lambda = (k-1)\rho \in P^+$. Then $\langle \lambda+\rho,\alpha\rangle = k\langle\rho,\alpha\rangle$, so $\dim L((k-1)\rho) = k^{|\Phi^+|}$. $\square$

3. Let $\omega_1, \dots, \omega_n$ be the fundamental weights of the complex semisimple Lie algebra $\mathfrak{g}$. Show that every finite dimensional simple $\mathfrak{g}$-representation occurs as a direct summand in a suitable tensor product (repetitions allowed) of the simple modules $L(\omega_1), \dots, L(\omega_n)$. (We call these simple modules, the fundamental representations of $\mathfrak{g}$.)

Solution: The fundamental weights are a $\mathbb{Z}_{\geq 0}$-basis of P^+, i.e., every $\lambda \in P^+$ can be written uniquely as $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ for $a_i \in \mathbb{Z}_{\geq 0}$. It is easy to see that the weight vectors of a tensor product $V \otimes U$ are $v \otimes u$ where v, u are weight vectors of V and U, respectively. Moreover, the weights of $V\otimes U$ are $\lambda+\mu$, λ a weight of V, μ a weight of U. Hence if λ, μ are highest weights of V, U, respectively, then $\lambda+\mu$ is a highest weight of $V\otimes U$. From complete reducibility, we have then that $L(\lambda+\mu)$ is a direct summand of $L(\lambda)\otimes L(\mu)$. By induction, it follows that $L(\lambda)$ is a direct summand of

$$L(\omega_1)^{\otimes a_1} \otimes \cdots \otimes L(\omega_n)^{\otimes a_n}.$$

In fact, one can do better, by replacing $L(\omega_i)^{\otimes a_i}$ with $S^{a_i}(L(\omega_i))$, where S^a is the a-symmetric power. $\square$

4. Let $\mathfrak{g} = sl(n, \mathbb{C})$.

(i) Use Weyl's dimension formula to show that $L(\omega_i) = \bigwedge^i V$, $1 \leq i \leq n-1$, where $V = \mathbb{C}^n$ is the standard representation.

(ii) Identify the adjoint representation in terms of the highest weight classification. (Why is the adjoint representation irreducible?)

Solution: (i) We think of $\mathfrak{h} = \{h \in \mathbb{C}^n \mid \operatorname{tr} h = 0\}$ and $\mathfrak{h}^* = \mathbb{C}^n/\langle \epsilon_1 + \cdots + \epsilon_n\rangle$. Then $\omega_i = \epsilon_1 + \cdots + \epsilon_i$, $1 \leq i \leq n-1$. From a previous problem sheet, or by a direct computation, $e_1\wedge\cdots\wedge e_i$ is a highest weight vector with weight ω_i of $\bigwedge^i V$. This means that $L(\omega_i)$ is a summand of $\bigwedge^i V$. We compute dimensions to show they are equal (there are many other ways to show equality). On the one hand, $\dim \bigwedge^i V = \binom{n}{i}$. We use Weyl's dimension formula to compute $L(\omega_i)$. We have $\rho = (n, n-1, \dots, 1)$, $\alpha_{ij} = \epsilon_i - \epsilon_j$, $i<j$, so $\langle \rho, \alpha_{ij}\rangle = j - i$, etc.

(ii) The adjoint representation if irreducible if and only if $\mathfrak{g}$ is simple, which is the case for $sl(n)$. The decomposition of $\mathfrak{g}$ into $\mathfrak{h}$-weight spaces is the Cartan decomposition, so the highest weight of $\mathfrak{g}$ is the highest root of $\mathfrak{g}$ with respect to $>$. For $\mathfrak{g} = sl(n)$, this is $\lambda = \epsilon_1 - \epsilon_n$. In terms of the fundamental weights

$$\epsilon_1 - \epsilon_n = \omega_1 + \omega_{n-1} \text{ in } \mathfrak{h}^*.$$

$\square$

5. Let $\mathfrak{g} = sl(3, \mathbb{C})$ and $L(\omega_1)$, $L(\omega_2)$ the two fundamental representations. Verify:

(i) $L(\omega_1)^* \cong L(\omega_2)$.

(ii) Konstant's multiplicity formula, and

(iii) Weyl's character formula for these two representations.

Solution: For $\mathfrak{g} = sl(3)$, $L(\omega_1) = V$, $L(\omega_2) = \bigwedge^2 V$, where $V = \mathbb{C}^3$.

(i) One can use linear algebra, e.g., $(\bigwedge^{n-i} V)^* \cong \bigwedge^i V$ if $\dim V = n$, or the highest weight classification, i.e.:

$$\omega_1 = \epsilon_1, \quad \omega_2 = \epsilon_1 + \epsilon_2 = -\epsilon_3 \text{ in } \mathfrak{h}^*,$$

and $w_0 = (13)$ for S_3, so $-w_0(\epsilon_1) = -\epsilon_3$; then use Problem 1(ii). In any case, $L(\omega_1)^* \cong L(\omega_2)$.

(ii) The weights of $L(\omega_1)$ are $\epsilon_1, \epsilon_2, \epsilon_3$, with multiplicity 1. The weights of $L(\omega_2)$ are $\epsilon_1 + \epsilon_2 = -\epsilon_3, \epsilon_1 + \epsilon_3 = -\epsilon_2, \epsilon_2 + \epsilon_3 = -\epsilon_1$, with multiplicity 1.

To verify Kostant's multiplicity formula for $L(\omega_1)$, write out the table:

w	det	$w \cdot \lambda$	$w \cdot \lambda - \mu, \mu = \epsilon_1$	K	$w \cdot \lambda - \mu, \mu = \epsilon_2$	K	$w \cdot \lambda - \mu, \mu = \epsilon_3$	K
$()$	1	$(1,0,0)$	$(0,0,0)$	1	$(1,-1,0)$	1	$(1,0,-1)$	2
(12)	-1	$(-1,2,0)$	$(-2,2,0)$	0	$(-1,1,0)$	0	$(-1,2,-1)$	0
(23)	-1	$(1,-1,1)$	$(0,-1,1)$	0	$(1,-2,1)$	0	$(1,-1,0)$	1
(13)	-1	$(-2,0,3)$	$(-3,0,3)$	0	$(-2,-1,3)$	0	$(-2,0,2)$	0
(123)	1	$(-2,2,1)$	$(-3,2,1)$	0	$(-2,1,1)$	0	$(-2,2,0)$	0
(132)	1	$(-1,-1,3)$	$(-2,-1,3)$	0	$(-1,-2,3)$	0	$(-1,-1,2)$	0

Then applying the formula we get that the multiplicities are 1, 1 and $2-1=1$, respectively. The case $L(\omega_2)$ is similar, in fact, it is just the table before to which we apply $-w_0$.

(iii) If we write the coordinates of $\mathfrak{h}$ as (x_1, x_2, x_3) with $x_1 + x_2 + x_3 = 0$, then

$$ch_{L(\omega_1)}(x_1,x_2,x_3) = p(x_1,x_2,x_3)/q(x_1,x_2,x_3), \text{ where}$$
$$p(x_1,x_2,x_3) = e^{4x_1+2x_2+x_3} - e^{2x_1+4x_2+x_3} - e^{4x_1+x_2+2x_3} - e^{x_1+2x_2+4x_3} + e^{x_1+4x_2+2x_3} + e^{2x_1+x_2+4x_3},$$
$$q(x_1,x_2,x_3) = e^{3x_1+2x_2+x_3} - e^{2x_1+3x_2+x_3} - e^{3x_1+x_2+2x_3} - e^{x_1+2x_2+3x_3} + e^{x_1+3x_2+2x_3} + e^{2x_1+x_2+3x_3},$$

and indeed, this equals $e^{x_1} + e^{x_2} + e^{x_3}$.

Similarly for $L(\omega_2)$. □

6. Let $\mathfrak{g} = sp(2n, \mathbb{C})$ realized as the space of matrices $X \in gl(2n, \mathbb{C})$ such that $X^t J + JX = 0$, where X^t is the transpose matrix, and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$; here I_n is the $n \times n$ identity matrix.

(i) Show that every $X \in \mathfrak{g}$ is of the form $X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$, where B and C are symmetric $n \times n$ matrices and A is an arbitrary $n \times n$ matrix.

(ii) Let $\mathfrak{h}$ be the subalgebra consisting of diagonal matrices. Determine the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ and the Cartan decomposition.

(iii) Choose the system of positive roots such that the corresponding root vectors lie in matrices of the form $\begin{pmatrix} A' & B \\ 0 & -A'^t \end{pmatrix}$, where A' is an upper triangular matrix and B is a symmetric matrix as before.

(iv) Determine the fundamental weights.

(v) Let $V = \mathbb{C}^{2n}$ be the standard representation of $\mathfrak{g}$ (which acts by matrix multiplication on column vectors). Show that V is an irreducible $\mathfrak{g}$-representation and it is in fact a fundamental representation.

(vi) Show that $\bigwedge^2 V$ decomposes as $W \bigoplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation and W is an irreducible (fundamental) representation.

(vii) For $sp(4, \mathbb{C})$, describe all the weights of the fundamental representations V and W and verify that the Weyl dimension formula holds.

(viii) In $sp(2n, \mathbb{C})$, show that the k-th fundamental representation is contained in $\bigwedge^k V$ and in fact it is precisely the kernel of the *contraction* map $\phi_k : \bigwedge^k V \to \bigwedge^{k-2} V$ defined by

$$\phi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k,$$

where Q is the skew-symmetric form defining $\mathfrak{g}$, i.e., $Q(v, u) = v^t J u$.

[*For this exercise, you may consult Section 16 in Fulton-Harris "Representation Theory", especially for the structural results on roots, Cartan decomposition etc.*]