Problem sheet 4 – Solutions 3/5/16.

① (i) L finite dimensional $\mathfrak{g}$-module

$L^* = \{f : L \to \mathbb{C}\}$ (linear maps) as a vector space. The action on L^* is given by $\boxed{(x\cdot f)(\ell) = -f(x\cdot \ell)}$, $x\in\mathfrak{g}$, $\ell\in L$, $f\in L^*$.

Define $A^\perp \subset L^*$ by $A^\perp = \{f\in L^* \mid f(a)=0 \ \forall a\in A\}$ for every submodule $A\subseteq L$. Then $A^\perp$ is a submodule for L^*. Moreover if A is proper in L, then $A^\perp$ is a proper submodule of L^*. Hence if L is not simple, L^* is not simple.

Conversely, if $B\subset L^*$ is a submodule, define $B^\perp \subset L$ by $B^\perp = \{\ell\in L \mid f(\ell)=0 \ \forall f\in B\}$. etc.

(ii) $L(\lambda)$, $\lambda\in P^+$ simple $\mathfrak{g}$-module. By (i), $L(\lambda)^*$ is also a simple f.d. $\mathfrak{g}$-module. It is easy to see that the weights of $L(\lambda)^*$ are precisely the negatives of the weights of $L(\lambda)$ (pick a basis of weight vectors of $L(\lambda)$ and take the dual basis in $L(\lambda)^*$; these are weight vectors of $L(\lambda)^*$). Moreover, we know that

$\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)}$ $\forall w\in W$, μ weight of $L(\lambda)$

This means that $\dim L(\lambda)_{w(\lambda)} = 1$ $\forall w\in W$. In particular, $\dim L(\lambda)_{w_0(\lambda)} = 1$, hence $\dim L(\lambda)^*_{-w_0(\lambda)} = 1$.

Notice $-w_0(\lambda)\in P^+$ iff $\lambda\in P^+$. This is because $-w_0(\alpha) > 0$ for all $\alpha>0$ (positive root). Moreover $-w_0(\lambda)$ is a maximal weight, otherwise $\exists\, \mu$ weight of $L(\lambda)$ (so $-\mu$ is a weight of $L(\lambda)^*$) s.t. $-\mu-(-w_0(\lambda))\in Q^+$. Apply $-w_0$ remembering that $-w_0(Q^+) = Q^+$ $\Rightarrow$ $w_0(\mu)-\lambda\in Q^+$. But $w_0(\mu)$ is also a weight of $L(\lambda)$, contradiction!

(iii) <u>Claim</u> 0 occurs as a weight of $L(\lambda)$ if and only if λ is a sum of positive roots. This follows immediately from the following remark:

By using $\mathfrak{sl}(2)$-theory for each $\mathfrak{sl}_\alpha$, α positive root, one can show that μ is a weight of $L(\lambda)$ if and only if $w(\mu) \leq \lambda$ for all $w\in W$ (one direction is immediate since μ weight $\Leftrightarrow$ $w(\mu)$ is a weight)

② Weyl's dimension formula:

$$\dim L(\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda+\rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \qquad \lambda \in P^+, \quad k \in \mathbb{N}$$

Take $\lambda = (k-1)\rho \in P^+$. Then $\langle \lambda+\rho, \alpha\rangle = k\langle \rho, \alpha\rangle$

$\Rightarrow \dim L((k-1)\rho) = k^{|\Phi^+|}$

③ $\omega_1, \dots, \omega_n$ are an $\mathbb{Z}_{\geq 0}$-basis of P^+, i.e., every $\lambda \in P^+$ can be written uniquely as $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ for $a_i \in \mathbb{Z}_{\geq 0}$. It is easy to see that the weight vectors of a tensor product $V \otimes U$ are precisely $v \otimes u$ where v, u are weight vectors of V and U, respectively. Moreover, the weights of $V \otimes U$ are $\lambda + \mu$, λ weight of V, μ weight of U. Hence if λ, μ are highest weights of V, U, resp., then $\lambda+\mu$ is a highest weight of $V \otimes U$. From complete reducibility then $L(\lambda+\mu)$ occurs as a summand of $L(\lambda) \otimes L(\mu)$. By induction, it follows that

$L(\lambda)$ is a direct summand of $L(\omega_1)^{\otimes a_1} \otimes \dots \otimes L(\omega_n)^{\otimes a_n}$

when $\lambda = a_1\omega_1 + \dots + a_n\omega_n$.

④ (i) We think of $\mathfrak{h} = \{h \in \mathbb{C}^n \mid \operatorname{tr} h = 0\}$ and $\mathfrak{h}^* = \mathbb{C}^n / \langle \varepsilon_1 + \dots + \varepsilon_n \rangle$. Then $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$ $\quad 1 \leq i \leq n-1$. From a previous problem sheet (or easily by a direct computation) $e_1 \wedge \dots \wedge e_i$ is a highest weight vector with weight ω_i of $\Lambda^i V$. This means that $L(\omega_i)$ is a summand of $\Lambda^i V$. We compute dimensions to show they are equal (there are many other ways to show equality). Use Weyl' dim formula to compute

$\dim \Lambda^i V = \binom{n}{i}$.

$\dim L(\omega_i)$: $\quad \rho = (n, n-1, \dots, 1)$, $\quad \alpha_{ij} = \varepsilon_i - \varepsilon_j \quad i<j$

so $\langle \rho, \alpha_{ij}\rangle = [n-(i-1)] - [n-(j-1)] = j-i$ etc

(ii) The adjoint repn is irreducible if and only if $\mathfrak{g}$ is simple, which is the case for $\mathfrak{g} = \mathfrak{sl}(n)$. The decomposition of $\mathfrak{g}$ into $\mathfrak{h}$-weight spaces is the Cartan decomposition, so the highest weight of $\mathfrak{g}$ is the highest root of $\mathfrak{g}$ w.r.t. $>$. For $\mathfrak{g} = \mathfrak{sl}(n)$ this is $\lambda = \varepsilon_1 - \varepsilon_n$. In terms of fundamental weights:

$$\varepsilon_1 - \varepsilon_n \underset{\text{in } \mathfrak{h}^*}{\equiv} \varepsilon_1 + (\varepsilon_1 + \dots + \varepsilon_{n-1}) = \omega_1 + \omega_{n-1}.$$

⑤ $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, $L(\omega_1) = V$, $L(\omega_2) = \Lambda^2 V$ where $V \simeq \mathbb{C}^3$

(i) One can use linear alg., e.g. $(\Lambda^{n-i} V)^* \simeq \Lambda^i V$ if $\dim V = n$ or the highest weights; i.e.,

$$\omega_1 = \varepsilon_1$$
$$\omega_2 = \varepsilon_1 + \varepsilon_2 \equiv -\varepsilon_3 \text{ in } \mathfrak{h}^*$$

and $w_0 = (13)$ for S_3, so $-w_0(\varepsilon_1) = -\varepsilon_3$

then use ①(ii). In any case $L(\omega_1)^* \simeq L(\omega_2)$.

(ii) The weights of $L(\omega_1) = V$ are $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$L(\omega_2) = \Lambda^2 V$ are $\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \varepsilon_2 + \varepsilon_3$ (which are $-\varepsilon_3$, $-\varepsilon_2$, $-\varepsilon_1$)

all have mult $= 1$.

$\rho = (3,2,1)$

$L(\omega_1)$: $\lambda = \omega_1 = (1,0,0) = \varepsilon_1$ $\quad w.\lambda = w(\lambda+\rho) - \rho \quad \lambda + \rho = (4\,2\,1)$

det	w	$w.\lambda$	$w.\lambda - \mu$; $\mu = \varepsilon_1$	K	$w.\lambda - \mu$; $\mu = \varepsilon_2$	K	$\mu = \varepsilon_3$	K
1	()	(1 0 0)	(0 0 0)	1	(1 -1 0)	1	(1 0 -1)	2
-1	(12)	(-1 2 0)	(-2 2 0)	0	(-1 1 0)	0	(-1 2 -1)	0
-1	(23)	(1 -1 1)	(0 -1 1)	0	(1 -2 1)	0	(1 -1 0)	1
-1	(13)	(-2 0 3)	(-3 0 3)	0	(-2 -1 3)	0	(-2 0 2)	0
1	(123)	(-2 2 1)	(-3 2 1)	0	(-2 1 1)	0	(-2 2 0)	0
1	(132)	(-1 -1 3)	(-2 -1 3)	0	(-1 -2 3)	0	(-1 -1 2)	0
				mult = 1		mult = 1		mult = 2 − 1 = 1

The case $L(\omega_2)$ is similar (in fact it's just the table before to which we apply $-w_0 = -(13)$)

(iii) If we write the coordinates of $\mathfrak{h}$ as (x_1, x_2, x_3) with $x_1+x_2+x_3=0$ then

$$ch_{L(\omega_1)}^{(x_1,x_2,x_3)} = \frac{e^{4x_1+2x_2+x_3} - e^{2x_1+4x_2+x_3} - e^{4x_1+x_2+2x_3} - e^{x_1+2x_2+4x_3} + e^{x_1+4x_2+2x_3} + e^{2x_1+x_2+4x_3}}{e^{3x_1+2x_2+x_3} - e^{2x_1+3x_2+x_3} - e^{3x_1+x_2+2x_3} - e^{x_1+2x_2+3x_3} + e^{x_1+3x_2+2x_3} + e^{2x_1+x_2+3x_3}}$$

as functions $\mathbb{C}^3 \to \mathbb{C}$.

$$\{ = e^{x_1} + e^{x_2} + e^{x_3} \}.$$

and similarly for $L(\omega_2)$.

(6) (i) Easy

(ii) $\mathfrak{h}$ is spanned by $H_i = E_{ii} - E_{n+i,n+i}$. Define the dual basis $\{\varepsilon_i\}$ of $\mathfrak{h}^*$ s.t. $\varepsilon_i(H_j) = \delta_{ij}$.

The roots are:
- $\varepsilon_i - \varepsilon_j$, $i \neq j$ with root vector $E_{ij} - E_{n+j,n+i}$
- $\varepsilon_i + \varepsilon_j$ $i \leq j$ — $E_{i,n+j} + E_{j,n+i}$
- $-\varepsilon_i - \varepsilon_j$ $i \leq j$ — $E_{n+i,j} + E_{n+j,i}$
- $2\varepsilon_i$ — $E_{i,n+i}$
- $-2\varepsilon_i$ — $E_{n+i,i}$

(iii) The choice of positive roots is. $\{\varepsilon_i - \varepsilon_j, i<j\} \cup \{\varepsilon_i + \varepsilon_j, i<j\} \cup \{2\varepsilon_i\}$.

(iv) The simple roots corresponding to (iii) are

$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$

. Dynkin diagram.

The fundamental weights (dual to the coroots! which are $H_1 - H_2, H_2 - H_3, \dots, H_{n-1} - H_n, H_n$) are:

~~$\omega_1 = \varepsilon$~~ $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, ..., $\omega_n = \varepsilon_1 + \dots + \varepsilon_n$.

(v) The ~~roots~~ weights of V are $\varepsilon_1, \dots, \varepsilon_n, -\varepsilon_1, \dots, -\varepsilon_n$ with weight vectors: $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$, respectively.

The highest weight is $\varepsilon_1 = \omega_1$. It's easy to see that V is irreducible, hence $\boxed{V = L(\omega_1)}$.

(One elegant way to see irreducibility is to realize that the Weyl group acts by permutations and sign changes in the usual coordinates. Hence the weights form a single W-orbit, so the representation must be irreducible.)

(vi) $\Lambda^2 V$ has weight vectors $e_k \wedge e_\ell$ with $1 \le k < \ell \le 2n$ and

weights		multiplicity	
$\varepsilon_i + \varepsilon_j$, $1 \le i < j \le n$		1	(*)
$\varepsilon_i - \varepsilon_j$, $1 \le i \ne j \le n$.		1	(*)
$-(\varepsilon_i + \varepsilon_j)$, $1 \le i < j \le n$.		1	(*)
0		n	

check: $4\binom{n}{2} + n = 4 \cdot \frac{n(n-1)}{2} + n = 2n(n-1) + n = n(2n-1)$

$= \binom{2n}{2}$ OK ✓

The vector $e_1 \wedge e_2$ is a highest weight vector with weight $\varepsilon_1 + \varepsilon_2$

$\Rightarrow \Lambda^2 V$ contains $L(\omega_2)$

Using the W-action, we can see that all the roots in (*) are in the W-orbit of ω_2 hence the only weights that may not be in $L(\omega_2)$ are 0-weights.

Using for example the Weyl dim formula, we may find (another way is to find the dim of the 0 weight space of $L(\omega_2)$)

$\dim L(\omega_2) = (n-1)(2n+1)$ (with $\rho = (n, n-1, \dots, 2, 1)$)

$\Rightarrow \dim \Lambda^2 V - \dim L(\omega_2) = 1$ and only the trivial has dim $= 1$

The trivial repn in $\Lambda^2 V$ is spanned by $e_1 \wedge e_{n+1} + e_2 \wedge e_{n+2} + \dots + e_n \wedge e_{2n}$

(vii) The weights of $W = L(\omega_2)$ are:

$\varepsilon_1+\varepsilon_2$, $\varepsilon_1-\varepsilon_2$, ~~$\varepsilon_1\ \varepsilon_2$~~ 0, $-\varepsilon_1+\varepsilon_2$, $-\varepsilon_1-\varepsilon_2$

all with multiplicity 1 in $sp(4)$.

Easy to verify Weyl's dimension.

(viii) Since $\varepsilon_1+\dots+\varepsilon_k = \omega_k$ is a highest weight of $\Lambda^k V$
it follows that $L(\omega_k)$ is a submodule of $\Lambda^k V$.
$\phi_k : \Lambda^k V \longrightarrow \Lambda^{k-2} V$. Notice that $\Lambda^{k-2} V$ does not have
the weight ω_k, hence $L(\omega_k) \subseteq \ker \phi_k$. So it is sufficient
to show $\ker \phi_k$ is irreducible. (One has to check that ϕ_k is
a $\mathfrak{g}$-hom. first to know that $\ker\phi_k$ is a $\mathfrak{g}$-module.)

One way to prove this is via restriction to $sl(n) \subset sp(2n)$

$$sl(n) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \right\}$$

as in Fulton–Harris, see attached.

As before, we denote this by $\Gamma_{a_1,\ldots,a_n}$:

$$\Gamma_{a_1,\ldots,a_n} = \Gamma_{a_1L_1+a_2(L_1+L_2)+\cdots+a_n(L_1+\cdots+L_n)}.$$

These exhaust all irreducible representations of $\mathfrak{sp}_{2n}\mathbb{C}$.

We can find the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,1,\ldots,0}$ with highest weight $L_1 + \cdots + L_k$ easily enough. Clearly, it will be contained in the kth exterior power $\wedge^k V$ of the standard representation. Moreover, we have a natural contraction map

$$\varphi_k : \wedge^k V \to \wedge^{k-2} V$$

defined by

$$\varphi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k$$

(see §B.3 of Appendix B for an intrinsic definition and explanation). Since the representation $\wedge^{k-2}V$ does not have the weight $L_1 + \cdots + L_k$, the irreducible representation with this highest weight will have to be contained in the kernel of this map. We claim now that conversely

Theorem 17.5. *For $1 \le k \le n$, the kernel of the map φ_k is exactly the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,0,1,0,\ldots,0}$ with highest weight $L_1 + \cdots + L_k$.*

PROOF. Clearly, it is enough to show that the kernel of φ_k is an irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$. We will do this by restricting to a subalgebra of $\mathfrak{sp}_{2n}\mathbb{C}$ isomorphic to $\mathfrak{sl}_n\mathbb{C}$, and using what we have learned about representations of $\mathfrak{sl}_n\mathbb{C}$.

To describe this copy of $\mathfrak{sl}_n\mathbb{C}$ inside $\mathfrak{sp}_{2n}\mathbb{C}$, consider the subgroup $G \subset \mathrm{Sp}_{2n}\mathbb{C}$ of transformations of the space $V = \mathbb{C}^{2n}$ preserving the skew form Q introduced in Lecture 16 and preserving as well the decomposition $V = \mathbb{C}\{e_1, \ldots, e_n\} \oplus \mathbb{C}\{e_{n+1}, \ldots, e_{2n}\}$. These can act arbitrarily on the first factor, as long as they do the opposite on the second; in coordinates, they are the matrices

$$G = \left\{ \begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix}, X \in \mathrm{GL}_n\mathbb{C} \right\}.$$

We have, correspondingly, a subalgebra

$$\mathfrak{s} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix}, A \in \mathfrak{sl}_n\mathbb{C} \right\} \subset \mathfrak{sp}_{2n}\mathbb{C}$$

isomorphic to $\mathfrak{sl}_n\mathbb{C}$.

Now, denote by W the standard representation of $\mathfrak{sl}_n\mathbb{C}$. The restriction of the representation V of $\mathfrak{sp}_{2n}\mathbb{C}$ to the subalgebra $\mathfrak{s}$ then splits

$$V = W \oplus W^*$$

into a direct sum of W and its dual; and we have, correspondingly,

$$\wedge^k V = \bigoplus_{a+b=k} (\wedge^a W \otimes \wedge^b W^*).$$

How does the tensor product $\wedge^a W \otimes \wedge^b W^*$ decompose as a representation of $\mathfrak{sl}_n\mathbb{C}$? We know the answer to this from the discussion in Lecture 15 (see Exercise 15.30): we have contraction maps

$$\Psi_{a,b} : \wedge^a W \otimes \wedge^b W^* \to \wedge^{a-1} W \otimes \wedge^{b-1} W^*;$$

and the kernel of $\Psi_{a,b}$ is the irreducible representation $W^{(a,b)} = \Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots}$ with (if, say, $a \le n - b$) highest weight $2L_1 + \cdots + 2L_a + L_{a+1} + \cdots + L_{n-b}$. The restriction of $\wedge^k V$ to $\mathfrak{s}$ is thus given by

$$\wedge^k V = \bigoplus_{\substack{a+b \le k \\ a+b \equiv k(2)}} W^{(a,b)}$$

and by the same token,

$$\mathrm{Ker}(\varphi_k) = \bigoplus_{a+b=k} W^{(a,b)}.$$

Note that the actual highest weight factor in the summand $W^{(a,b)} \subset \mathrm{Ker}(\varphi_k) \subset \wedge^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n}. \end{aligned}$$

Exercise 17.6. Show that more generally the highest weight vector in any summand $W^{(a,b)} \subset \wedge^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n} \wedge \left(\sum (e_i \wedge e_{n+i})\right)^{(k-a-b)/2}. \end{aligned}$$

By the above, any subspace of $\mathrm{Ker}(\varphi_k)$ invariant under $\mathfrak{sp}_{2n}\mathbb{C}$ must be a direct sum, over a subset of pairs (a, b) with $a + b = k$, of subspaces $W^{(a,b)}$. But now (supposing for the moment that $k < n$) we observe that the element

$$Z_{a,n-b} = E_{2n-b,a} + E_{n+a,n-b} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries the vector $w^{(a,b)}$ into $w^{(a-1,b+1)}$ and, likewise,

$$Y_{a+1,n-b+1} = E_{a+1,2n-b+1} + E_{n-b+1,n+a+1} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries $w^{(a,b)}$ to $w^{(a+1,b-1)}$. In case $a + b = k = n$, we see similarly that

$$V_a = E_{n+a,a} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries the vector $w^{(a,b)}$ into $w^{(a-1,b+1)}$, and

$$U_{a+1} = E_{a+1,n+a+1} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries $w^{(a,b)}$ to $w^{(a+1,b-1)}$. Thus, any representation of $\mathfrak{sp}_{2n}\mathbb{C}$ contained in $\mathrm{Ker}(\varphi_k)$ and containing any one of the factors $W^{(a,b)}$ will contain them all, and we are done. □