REPRESENTATION THEORY OF SEMISIMPLE LIE ALGEBRAS

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Contents

The representation theory of semisimple Lie algebras plays a central role in modern mathematics with motivation coming from many areas of mathematics and physics, for example, the Langlands program. The methods involved in the theory are diverse and include remarkable interactions with algebraic geometry, as in the proofs of the Kazhdan-Lusztig and Jantzen conjectures.

The course will cover the basics of finite dimensional representations of semisimple Lie algebras (e.g., the Cartan-Weyl highest weight classification, Weyl's character and dimension formulas) in the framework of the larger Bernstein-Gelfand-Gelfand category $\mathcal{O}$.

These notes are based on J. Bernstein's "Lectures on Lie algebras" [Be], supplemented by material from [Di], [Hu1], [Hu2], [FH].

1. The universal enveloping algebra of a Lie algebra

1.1. Lie algebras. Let k be a field.

Definition 1.1. *A Lie algebra $\mathfrak{g}$ over k is a k-vector space with a bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the Lie bracket) that satisfies the following identities:*

(1) *(alternating) $[x, x] = 0$ for all $x \in \mathfrak{g}$. (When fc $k \neq 2$, this is equivalent with "skew-symmetry": $[x, y] = -[y, x]$, $x, y \in \mathfrak{g}$.)*
(2) *(Jacobi identity): for all $x, y, z \in \mathfrak{g}$,*

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

A Lie algebra is a non-associative algebra, and the Jacobi identity replaces the associativity condition. There are a few basic examples to keep in mind.

Example 1.2. (1) *Let A be an associative algebra. Then we may define a Lie algebra structure on $\mathfrak{g} = A$ by setting the bracket to equalded the commutator in A; $[x, y] = xy - yx$. One verifies immediately that the Jacobi identity is satisfied because of the associativity of the multiplication in A.*

(a) *Let V be a k-vector space and $A = \mathrm{End}_k(V)$. Applying the construction above to this setting, we obtain the Lie algebra $\mathfrak{gl}(V)$ whose elements are the endomorphisms of V and the bracket is the commutator.*
(b) *If V is finite dimensional and we fix a basis of V, we may identify $V \cong k^n$ and $\mathrm{End}_k(V)$ with $n \times n$ matrices with coefficients in k. Then the Lie algebra is $\mathfrak{gl}(n, k)$, the general linear Lie algebra of $n \times n$ matrices with the Lie bracket given by the commutator.*

[0]Notes for Oxford's Part C course C2.3 "Representations of semisimple Lie algebras".

(2) *Let V be a k-vector space, and consider $\mathfrak{sl}(V) = \{x : V \to V \mid \mathrm{tr}(x) = 0\}$ with the Lie bracket given by the commutator in $\mathfrak{gl}(V)$. (Recall that the commutator of any two linear maps has trace 0.) If V is finite dimensional and we fix a basis as before, we obtain the special linear Lie algebra $\mathfrak{sl}(n, k)$ of $n \times n$ matrices of trace 0. Notice that this algebra is not an example of (1): in order to define the bracket in $\mathfrak{sl}(V)$, we invoke the commutator in a larger algebra, $\mathfrak{gl}(V)$. In fact, one may prove that in general there is no associative algebra A such that $\mathfrak{sl}(2)$ is isomorphic with the Lie algebra obtained from A via the construction in (1).*

(3) *(Classical Lie algebras) Let V be a finite dimensional vector space over k and $B : V \times V \to k$ be a bilinear form. Define*

$$\mathrm{Der}(B) = \{x \in \mathfrak{gl}(V) \mid B(xu, v) + B(u, xv) = 0, \text{ for all } u, v \in V\}. \tag{1.1.1}$$

Thi is a Lie subalgebra of $\mathfrak{gl}(V)$, consisting of the linear maps that preserve B. Suppose that B is nondegenerate.

(a) *If B is symmetric, we obtain the orthogonal Lie algebra with respect to B, denoted by $\mathfrak{so}(V, B)$. When $k = \mathbb{C}$, all nondegenerate symmetric bilinear forms are equivalent, hence there is only one (up to isomorphism) orthogonal Lie algebra over $\mathbb{C}$. On the other hand, if $k = \mathbb{R}$, then the nondegenerate symmetric bilinear forms are classified by their signatures, and so are the orthogonal Lie algebras over $\mathbb{R}$.*

(b) *If B is skew-symmetric, we obtain the symplectic Lie algebra with respect to B, denoted by $\mathfrak{sp}(V, B)$. Since B is nondegenerate, $\dim V$ must be even. Recall that, unlike the case of symmetric bilinear forms, the classification of skew-symmetric bilinear forms is independent of the field. In particular, there exists only one (up to equivalence) nondegenerate skew-symmetric bilinear form and thus, only one symplectic Lie algebra (up to isomorphism).*

(4) *Let $\mathfrak{g} = \mathfrak{gl}(n, k)$ be the general linear Lie algebra. We denote by $\mathfrak{n}^+$, $\mathfrak{h}$, $\mathfrak{n}^-$ the Lie subalgebras of strictly upper triangular matrices, diagonal matrices, and strictly lower triangular matrices, respectively. The vector space decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ will play an important role in the theory. We emphasize that this is not a Lie algebra decomposition.*

1.2. Lie algebra representations.

Definition 1.3. *A representation of the Lie algebra $\mathfrak{g}$ over k is a k-vector space V together with a linear map $\rho : \mathfrak{g} \to \mathrm{End}_k(V)$ (the action) such that*

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x), \text{ for all } x, y \in \mathfrak{g}. \tag{1.2.1}$$

An equivalent way is to say that the map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a homomorphism of Lie algebras.

Example 1.4. (1) *Let $\mathfrak{g} = \mathfrak{gl}(V)$ act on V in the usual way, i.e., the map ρ is the identity. In terms of matrices, if $\mathfrak{g} = \mathfrak{gl}(n)$ and $V = k^n$, then the action is just matrix multiplication: an $n \times n$ matrix times a column vector.*

(2) *If $\mathfrak{g}$ is any Lie algebra, the adjoint representation is $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathrm{ad}(x)(y) = [x, y]$, for all $x, y \in \mathfrak{g}$. The fact that this is a representation is equivalent with the Jacobi identity.*

(3) *If $\mathfrak{g}$ is any Lie algebra, the trivial representation is the one dimensional representation $\rho_0 : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C})$, given by $\rho_0(x) = 0$ for all $x \in \mathfrak{g}$. More generally, if (ρ, V) is any $\mathfrak{g}$-representation, we write*

$$V^{\mathfrak{g}} = \{v \in V \mid \rho(x)v = 0, \text{ for all } x \in \mathfrak{g}\}. \tag{1.2.2}$$

This is a subrepresentation of V consisting of all the copies of the trivial representation that occur in V.

If (ρ, V) is a representation of $\mathfrak{g}$, we will often write $x \cdot v$ in place of $\rho(x)(v)$ for the action, $x \in \mathfrak{g}$, $v \in V$. Later in the course we will specialize to certain types of representations.

1.3. **Tensor products.** Recall that the tensor product of two k-vector spaces U, V is a k-vector space $U \otimes V$ which satisfies a bilinearity property:

(1) $(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$, $u_1, u_2 \in U$, $v \in V$;
(2) $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$, $u \in U$, $v_1, v_2 \in V$;
(3) $(\lambda u) \otimes v = u \otimes (\lambda v) = \lambda(u \otimes v$, $\lambda \in k$, $u \in U$, $v \in V$.

A typical element in $U \otimes V$ is $\sum_{i=1}^n u_i \otimes v_i$, where $u_i \in U$ and $v_i \in V$. If $\{e_i \mid i \in I\}$ is a basis of U and $\{f_j \mid j \in J\}$ is a basis of V, then $\{e_i \otimes f_j \mid i \in I,\ j \in J\}$ is a basis of $U \otimes V$. In particular, $\dim_k(U \otimes V) = \dim_k U \cdot \dim_k V$.

More generally, if $V_1, \dots, V_n$ are k-vector spaces, we may define recursively the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_k$. Since the tensor product of vector spaces is associative, we can ignore the order in which we construct this tensor product, e.g., $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$.

In particular, we can speak about the n-fold tensor product of a vector space V, $T^n(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$.

If U and V are $\mathfrak{g}$-representations, then we define an action of $\mathfrak{g}$ on $U \otimes V$ by:

$$x \cdot (u \otimes v) = (x \cdot u) \otimes v + u \otimes (x \cdot v), \quad x \in \mathfrak{g},\ u, v \in V. \tag{1.3.1}$$

One verifies easily that this is indeed a Lie algebra action:

$$\begin{aligned} x \cdot (y \cdot (u \otimes v)) &= x \cdot ((y \cdot u) \otimes v + u \otimes (y \cdot v)) \\ &= (x \cdot (y \cdot u)) \otimes v + (y \cdot u) \otimes (x \cdot v) + (x \cdot u) \otimes (y \cdot v) + u \otimes (x \cdot (y \cdot v)). \end{aligned}$$

Writing the similar equation for $y \cdot (x \cdot (u \otimes v))$ and substracting, we see that the middle terms cancel and we find:

$$[x, y] \cdot (u \otimes v) = [x, y] \cdot u \otimes v + u \otimes [x, y] \cdot v.$$

We can extend this definition to an action on tensor products $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, as the sum of actions on one component at a time.

Definition 1.5. *V be a k-vector space. Set*

$$T(V) = \sum_{n \geq 0} T^n(V). \tag{1.3.2}$$

By convention, $T^0(V) = k$. Endow $T(V)$ with the multiplication given by the tensor product: $T^i(V) \times T^j(V) \to T^{i+j}(V)$, $(x, y) \mapsto x \otimes y$. This makes $T(V)$ into an associative k-algebra with unity, called the tensor algebra of V. (The element 1 comes from $k = T^0(V)$.)

If V is a $\mathfrak{g}$-representation, then $T(V)$ is a $\mathfrak{g}$-representation with the action on each $T^n(V)$ as before.

The *symmetric algebra* $S(V)$ of V is defined as the quotient of $T(V)$ by the two-sided ideal generated by all elements $x \otimes y - y \otimes x$, $x, y \in V$. Since the generators are homogeneous, $S(V)$ is also automatically graded $S(V) = \oplus_{n \geq 0} S^n(V)$. If $\{x_i \mid i \in I\}$ is a basis of V where $(I, \leq)$ is an ordered set, then a basis of $S^n(V)$ is given by $\{x_{i_1} x_{i_2} \dots x_{i_n} \mid i_1 \leq i_2 \leq \cdots \leq i_n,\ i_j \in I\}$.

The *exterior algebra* $\bigwedge V$ of V is defined as the quotient of $T(V)$ by the two-sided ideal geberated by all the elements $x \otimes x$, $x \in V$. Also $\bigwedge V$ is a graded algebra. If $\{x_i \mid i \in I\}$ is a basis of V where $(I, \leq)$ is an ordered set, then a basis of $\bigwedge^n V$ is given by $\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} \mid i_1 < i_2 < \cdots < i_n,\ i_j \in I\}$. In particular, if $n > \dim V$, then $\bigwedge^n V = 0$.

When V is a $\mathfrak{g}$-module, the symmetric and exterior algebras $S(V)$, $\bigwedge V$ inherit a $\mathfrak{g}$-action from $T(V)$.

Example 1.6. *Suppose the characteristic of the field k is not 2. We can decompose $V \otimes V$ as a direct sum:*

$$V \otimes V = S^2(V) \oplus \bigwedge^2 V,$$

where we embed $S^2(V)$ into $V \otimes V$ via:

$$xy \mapsto \frac{1}{2}(x \otimes y + y \otimes x),$$

and we embed $\bigwedge^2 V$ into $V \otimes V$ via:

$$x \wedge y \mapsto \frac{1}{2}(x \otimes y - y \otimes x).$$

If V is a $\mathfrak{g}$-representation, then this decomposition of $\mathfrak{g}$-invariant, in other words, it is a decomposition as $\mathfrak{g}$-representations.

1.4. **The universal enveloping algebra: definition.** Let $\mathfrak{g}$ be a Lie algebra. The goal is to assign to $\mathfrak{g}$ an associative k-algebra with 1 such that the representation theory of $\mathfrak{g}$ is equivalent with the representation theory of this associative algebra.

Definition 1.7. *Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. Let J be the two-sided ideal of $T(\mathfrak{g})$ generated by all the elements $x \otimes y - y \otimes x - [x, y]$, with $x, y \in \mathfrak{g}$. The universal enveloping algebra of $\mathfrak{g}$ is the associative unital k-algebra:*

$$U(\mathfrak{g}) = T(\mathfrak{g})/J. \tag{1.4.1}$$

There is a canonical linear map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ obtained by composing the identity map $\mathfrak{g} \to T^1(\mathfrak{g})$ with the quotient map $T(\mathfrak{g}) \to U(\mathfrak{g})$.

The adjective "universal" is motivated by the following universal property whose proof is straightforward.

Lemma 1.8. *Let A be an associative unital algebra together with a linear map $\tau : \mathfrak{g} \to A$ such that $\tau(x)\tau(y) - \tau(y)\tau(x) = \tau([x, y])$, for all $x, y \in \mathfrak{g}$. There exists one and only one algebra homomorphism $\tau' : U(\mathfrak{g}) \to A$ such that $\tau'(1) = 1$ and*

$$\tau' \circ \iota = \tau.$$

In particular, notice that the lemma says that every Lie algebra representation of $\mathfrak{g}$ can be lifted to a representation of the associative algebra $U(\mathfrak{g})$. Indeed, if we have $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a Lie algebra representation, take $A = \mathrm{End}_k(V)$ and $\tau = \rho$ and the claim follows from the universal property. Conversely, given any representation of $U(\mathfrak{g})$, we obtain a Lie algebra representation of $\mathfrak{g}$ by composing with the canonical map ι. Therefore, Lie algebra representations of $\mathfrak{g}$ are the same thing as representations of $U(\mathfrak{g})$.

Example 1.9. *If $\mathfrak{g}$ is a commutative Lie algebra (so the bracket is identically zero), then $U(\mathfrak{g}) = S(\mathfrak{g})$, the symmetric algebra generated by (the vector space) $\mathfrak{g}$.*

1.5. **Filtration by degree and the associated graded algebra.** We assume from now on that k has characteristic 0 and that $\mathfrak{g}$ is a finite dimensional Lie algebra over k.

The tensor algebra $T(\mathfrak{g})$ has a natural filtration by degree via the subspaces $T_n(\mathfrak{g}) = \sum_{i=0}^n T^i(\mathfrak{g})$. Let $U_n(\mathfrak{g})$ denote the image of $T_n(\mathfrak{g})$ in $U(\mathfrak{g})$. Then $\{U_n(\mathfrak{g})\}$ is a filtration by subspaces of $U(\mathfrak{g})$:

$$U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \cdots \subset U_n(\mathfrak{g}) \subset \ldots, \quad U(\mathfrak{g}) = \cup_{n \geq 0} U_n(\mathfrak{g}). \tag{1.5.1}$$

Definition 1.10. *The associated graded algebra $\mathrm{gr}\, U(\mathfrak{g})$ is defined as follows. As a vector space, it equals $G^0 \oplus G^1 \oplus G^2 \oplus \ldots$, where $G^n = U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$. (The convention is that $U_{-1}(\mathfrak{g}) = 0$.) The multiplication in $U(\mathfrak{g})$ defines bilinear maps $G^n \times G^m \to G^{n+m}$ and therefore a multiplication on $\mathrm{gr}\, U(\mathfrak{g})$. This makes $\mathrm{gr}\, U(\mathfrak{g})$ into an associative unital algebra.*

Lemma 1.11. *Let $a_1, a_2, \ldots, a_m$ be elements of $\mathfrak{g}$ and let σ be a permutation of $\{1, \ldots, n\}$. Then*

$$\iota(a_1)\iota(a_2)\ldots\iota(a_m) - \iota(a_{\sigma(1)})\iota(a_{\sigma(2)})\ldots\iota(a_{\sigma(m)}) \in U_{m-1}(\mathfrak{g}).$$

Proof. Since every permutation can be written as a product of simple transpositions, it is sufficient to prove the claim when $\sigma = (j, j+1)$. But in this case, the claim is immediate from the identity

$$\iota(a_j)\iota(a_{j+1}) - \iota(a_{j+1})\iota(a_j) = \iota([a_j, a_{j+1}]),$$

which shows the drop in degree by one. □

As a consequence, we see that

Lemma 1.12. $\mathrm{gr}\, U(\mathfrak{g})$ *is commutative.*

Proof. This is immediate from the previous lemma since in $\mathrm{gr}\, U(\mathfrak{g})$, we consider succesive quotients. □

Fix an ordered basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$. Denote the image of this basis in $U(\mathfrak{g})$ by $\{y_1, \ldots, y_n\}$. For every finite sequence $I = (i_1, \ldots, i_m)$ of integers between 1 and m, let $y_I = y_{i_1} y_{i_2} \ldots y_{i_m} \in U(\mathfrak{g})$.

Lemma 1.13. *The vector space $U_m(\mathfrak{g})$ is generated by y_I for all increasing sequences I of length at most m.*

Proof. The claim is clear without the adjective "increasing". Lemma 1.11 shows that indeed we may take only increasing sequences. □

We wish to show that the canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is injective and to understand the structure of the commutative algebra $\operatorname{gr} U(\mathfrak{g})$. The main technical result that we need is next.

Let $P = \mathbb{C}[z_1, \dots, z_n]$ be the algebra of polynomials in n indeterminates $z_1, \dots, z_n$. Let P_m denote the subspace of polynomials of total degree less than or equal to m. If $I = (i_1, \dots, i_m)$ is a sequence of integers, denote z_I as before. It will be convenient to use the notation $i \le I$ whenever $i \le i_k$ for all $k = 1, \dots, m$.

Lemma 1.14 (Dixmier)**.** *For every $m \ge 0$, there exists a unique linear map $f_m : \mathfrak{g} \otimes P_m \to P$ such that:*

(A_m) *$f_m(x_i \otimes z_I) = z_i z_I$ for $i \le I$, $z_I \in P_m$;*
(B_m) *$f_m(x_i \otimes z_I) - z_i z_I \in P_k$ for $z_I \in P_k$, $k \le m$;*
(C_m) *$f_m(x_i \otimes f_m(x_j \otimes z_J)) = f_m(x_j \otimes f_m(x_i \otimes z_J)) + f_m([x_i, x_j] \otimes z_J)$, for $z_J \in P_{m-1}$. (The terms in (C_m) make sense by virtue of (B_m).)*

Moreover, the restriction of f_m to $\mathfrak{g} \otimes P_{m-1}$ is f_{m-1}.

Notice that this lemma simply says that there exists a canonical representation of $U(\mathfrak{g})$ on the space of polynomials P.

Proof. [1] (Dixmier, page 68) The last assertion follows from the uniqueness of the maps f_m since the restriction of f_m to $\mathfrak{g} \otimes P_{m-1}$ satisfies (A_{m-1}), (B_{m-1}), and (C_{m-1}).

The proof is by induction on m. For $m = 0$, set $f_0(x_i \otimes 1) = z_i$, which by (A_0) is the only possibility. Then (B_0) and (C_0) are also satisfied. Now assume the existence and uniqueness of f_{m-1}. We need to prove that f_{m-1} has one and only one extension f_m to $\mathfrak{g} \otimes P_m$ satisfying (A_m), (B_m), and (C_m).

Let I be an increasing sequence of m integers between 1 and n. We define $f_m(x_i \otimes z_I)$. If $i \le I$, then (A_m) determines it. Otherwise, write $I = (j, J)$ with $j < i$ and $j \le J$. Then

$$\begin{aligned} f_m(x_i \otimes z_I) &= f_m(x_i \otimes f_{m-1}(x_j \otimes z_J)) \quad \text{by } (A_{m-1}) \\ &= f_m(x_j \otimes f_{m-1}(x_i \otimes z_J)) + f_{m-1}([x_i, x_j] \otimes z_J) \quad \text{by } (C_{m-1}). \end{aligned}$$

Moreover, by (B_{m-1}): $f_{m-1}(x_i \otimes z_J) = z_i z_J + w$ for some $w \in P_{m-1}$. Thus

$$\begin{aligned} f_m(x_i \otimes f_{m-1}(x_j \otimes z_J)) &= z_j z_i z_J + f_{m-1}(x_j \otimes w) \quad \text{from } (A_m) \\ &= z_i z_I + f_{m-1}(x_j \otimes w). \end{aligned}$$

This defines f_m uniquely, and with this definition, f_m satisfies (A_m) and (B_m). It remains to prove that f_m also satisfies (C_m). Condition (C_m) is satisfied by construction if $j \le i$ and $j \le J$. Using $[x_j, x_i] = -[x_i, x_j]$, it is also satisfied is $i \le j$ and $i \le J$. So (C_m) holds if $i \le J$ or $j \le J$.

Otherwise, write $J = (k, K)$ with $k \le K$, $k < i$, $k < j$. To simplify notation, write $xz := f_m(x \otimes z)$, for $x \in \mathfrak{g}$ and $z \in P_m$. By induction, we have:

$$x_j z_J = x_j(x_k z_K) = x_k(x_j z_K) + [x_j, x_k] z_K.$$

Now $x_j z_K = z_j z_K + w$ for some $w \in P_{m-2}$. Apply (C_m) to $x_i(x_k(z_j z_K))$ since $k \le K$ and $k < j$ and to $x_i(x_k w)$ from the induction hypothesis, hence to $x_i(x_k(x_j z_K))$. Then:

$$x_i(x_j z_J) = x_k(x_i(x_j z_K)) + [x_i, x_k](x_j z_K) + [x_j, x_k](x_i z_K) + [x_i, [x_j, x_k]] z_K.$$

Interchange i and j and perform the cancellations to get

$$x_i(x_j z_J) - x_j(x_i z_J) = [x_i, x_j] x_k z_K = [x_i, x_j] z_K.$$

This proves the claim.

□

Proposition 1.15. *The set $\{y_I \mid I \text{ increasing sequence}\}$ is a basis of $U(\mathfrak{g})$.*

Proof. Lemma 1.14 can be rephrased as saying that there is a representation $\rho : \mathfrak{g} \to \operatorname{End}(P)$ such that $\rho(x_i) z_I = z_i z_I$ for all $i \le I$. By the universal property, there exists a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to \operatorname{End}(P)$ such that

$$\phi(y_i) z_I = z_i z_I, \text{ for all } i \le I.$$

From this, we deduce recursively that, if I is an increasing sequence, then $\phi(y_I)1 = z_I$. Since $\{z_I\}$ are linearly independent in P, it follows that $\{y_I\}$ is a linearly independent set in $U(\mathfrak{g})$. But we already know that it is also a generating set, hence a basis. □

[1] This proof is non-examinable.

Corollary 1.16. *The canonical map* $\iota : \mathfrak{g} \to U(\mathfrak{g})$ *is injective.*

Proof. Clear from Proposition 1.15. □

In light of this result, from now on identify $\mathfrak{g}$ with its image in $U(\mathfrak{g})$ and drop ι (and the y's) from notation.

Corollary 1.17. *Let* $(x_1, \dots, x_n)$ *be an ordered basis of* $\mathfrak{g}$*. Then* $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$*,* $k_i \in \mathbb{N}$ *form a basis of* $U(\mathfrak{g})$*.*

Proof. This is just a rephrasing of Proposition 1.15. □

Since $\iota : \mathfrak{g} \to \operatorname{gr} U(\mathfrak{g})$ ($\mathfrak{g} \cong G^1$) is an injection, we can uniquely extend it to a canonical homomorphism $\iota : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$. (Both algebras are commutative!) Clearly, $\iota(S^n(\mathfrak{g})) \subset G^n$.

Theorem 1.18 (Poincaré-Birkhoff-Witt Theorem)**.** *The canonical homomorphism* $\iota : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$ *is an isomorphism of graded algebras.*

Proof. Let $(x_1, \dots, x_n)$ be an ordered basis of $\mathfrak{g}$. For $\underline{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, let $|\underline{k}| = k_1 + \dots + k_n$ and denote $X^{\underline{k}} = x_1^{k_1} \dots x_n^{k_n} \in S(\mathfrak{g})$ and $x^{\underline{k}} = x_1^{k_1} \dots x_n^{k_n} \in U(\mathfrak{g})$. Let $\overline{x}^{\underline{k}} = x_1^{k_1} \dots x_n^{k_n}$ denote the image of $x^{\underline{k}}$ in $G^{|\underline{k}|}$.

Since $\phi(X^{\underline{k}} = \overline{x}^{\underline{k}}$ and $\{\overline{x}^{\underline{k}}\}$ form a basis of $G^{|\underline{k}|}$ (since $\{x^{\underline{k}}\}$ is a basis of $U_{|\underline{k}|}(\mathfrak{g})$ by Corollary 1.17), it follows that ϕ is bijective. □

Remark 1.19. *The PBW theorem allows us to identify canonically* $\operatorname{gr} U(\mathfrak{g})$ *with* $S(\mathfrak{g})$*.*

Let $\mathsf{symm}' : S(\mathfrak{g}) \to T(\mathfrak{g})$ be the symmetrizing map:

$$x_1 x_2 \dots x_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(m)}. \tag{1.5.2}$$

Denote $\mathsf{symm} : S(\mathfrak{g}) \to U(\mathfrak{g})$ the composition of the map symm' with the projection onto $U(\mathfrak{g})$. The results we have proved so far show that symm is an isomorphism of linear spaces. We emphasize that it is not an isomorphism of algebras! This is obvious, since $S(\mathfrak{g})$ is commutative, but $U(\mathfrak{g})$ is not.

Remark 1.20. *One can show that* symm *is in fact an isomorphism of* $\mathfrak{g}$*-modules (homework). If we accept this, then by taking the* $\mathfrak{g}$*-invariants (the copies of the trivial representation), we obtain a linear isomorphism* $\mathsf{symm} : S(\mathfrak{g})^{\mathfrak{g}} \to Z(\mathfrak{g})$*, where* $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ *is the center of* $U(\mathfrak{g})$*. But again,* symm *is just an isomorphism of linear spaces and not of algebras in general, even though now both algebras are commutative.*

1.6. **The principal anti-automorphism of $U(\mathfrak{g})$.** The principal anti-automorphism of $U(\mathfrak{g})$ is an algebra anti-automorphism ${}^T : U(\mathfrak{g}) \to U(\mathfrak{g})$ defined by

$$(x_1 x_2 \dots x_n)^T = (-1)^n x_n x_{n-1} \dots x_1. \tag{1.6.1}$$

It is the unique anti-automorphism of $U(\mathfrak{g})$ such that $x^T = -x$ for all $x \in \mathfrak{g}$.

If ρ is a Lie algebra representation of $\mathfrak{g}$, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we have the contragredient representation $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$. The mapping

$$u \mapsto {}^t\rho(u^T), \quad u \in U(\mathfrak{g}),$$

is a representation of $U(\mathfrak{g})$ which extends ρ^*.

2. Representations of $\mathfrak{sl}(2)$

From now on, the field is assumed to be $k = \mathbb{C}$. In this section, we study finite dimensional representations of $\mathfrak{g} = \mathfrak{sl}(2)$.

2.1. **Weights and weight vectors.** The Lie algebra $\mathfrak{sl}(2)$ consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a + d = 0$ (trace zero). The standard basis of $\mathfrak{sl}(2)$ is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.1.1}$$

The relations between the basis elements are

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Lemma 2.1. *The following identities hold in $U(\mathfrak{sl}(2))$:*

$$[h, e^k] = 2ke^k,\ [h, f^k] = -2kf^k,\ [e, f^k] = kf^{k-1}(h - (k-1)),$$

for all integers $k \geq 1$.

Proof. Straightforward, by induction on k. □

An important role in the representation theory of $\mathfrak{sl}(2)$ is played by the Casimir element. Define

$$C = h^2 + 2h + 4fe \in U(\mathfrak{sl}(2)). \tag{2.1.2}$$

We will refer to this element as the Casimir element of $U(\mathfrak{sl}(2))$, but it is unique only up to a scalar multiple, as we will see when we discuss the general theory for a semisimple Lie algebra.

Lemma 2.2. *The Casimir element C belongs to the center of $U(\mathfrak{sl}(2))$.*

Proof. By the results of the previous section, $U(\mathfrak{sl}(2))$ is generated by e, h, f, therefore it is sufficient to check that C commutes with e and f (as h is the commutator of e and f). This is a direct calculation. For example

$$[C, e] = [h^2, e] + 2[h, e] + 4[fe, e],$$

and $[h^2, e] = h^2e - eh^2 = h([h, e] + eh) - ([e, h] + he)h = 2he + 2eh = 4he - 4e$. Moreover, $[h, e] = 2e$ and $[fe, e] = fe^2 - efe = ([f, e] + ef)e - efe = -he$. This shows that the sum above is zero indeed. □

Definition 2.3. *Let V be an $\mathfrak{sl}(2)$-module. A vector $v \in V$ is called a weight vector if it is an eigenvector for the action of h, i.e., $h \cdot v = \lambda v$ for some $\lambda \in \mathbb{C}$. If $v \neq 0$ is a weight vector, we call the corresponding eigenvalue λ a weight. Denote*

$$V_\lambda^{ss} = \{v \in V \mid (h - \lambda)v = 0\}, \quad V_\lambda = \{v \in V \mid \exists N > 0 \text{ such that } (h - \lambda)^N v = 0\},$$

and call them the λ-weight space and the generalized λ-weight space, respectively.

Clearly $V_\lambda^s s \subseteq V_\lambda$. If $V_\lambda^{ss} = V_\lambda$ for all λ, we say that h acts semisimply on V.

Lemma 2.4. *Let V be an $\mathfrak{sl}(2)$-module. Then:*

(1) $e \cdot V_\lambda \subseteq V_{\lambda+2}$;
(2) $f \cdot V_\lambda \subseteq V_{\lambda-2}$.

The same formulas hold with V_λ^{ss} in place of V_λ.

Proof. Suppose $v \in V_\lambda$ is given. Then there exists $N > 0$ such that $(h - \lambda)^N v = 0$. Notice that in $U(\mathfrak{sl}(2))$, $(h - 2)e = eh$ which means that $(h - 2)^j e = eh^j$ for all j. Then $(h - \lambda - 2)^N e \cdot v = [(h - 2) - \lambda]^N e \cdot v = e(h - \lambda)^N v = 0$, which means that $e \cdot v \in V_{\lambda+2}$.

The case of V_λ^{ss} is when $N = 1$. The statement about f is completely similar. □

Definition 2.5. *A vector $v \neq 0$ in V is called a highest weight vector if $v \in V_\lambda^{ss}$ for some λ and $e \cdot v = 0$.*

If V is a finite dimensional $\mathfrak{sl}(2)$-module, then Lemma 2.4 implies that highest weight vectors do exist.

Lemma 2.6. *Let V be an $\mathfrak{sl}(2)$-module and $v \in V$ a highest weight vector of weight λ. Consider the sequence of vectors $v_0 = v$, $v_k = f^k \cdot v$, for $k \geq 0$. Then:*

(a) *$h \cdot v_k = (\lambda - 2k)v_k$, $e \cdot v_0 = 0$, $e \cdot v_{k+1} = (k + 1)(\lambda - k)v_k$, $f \cdot v_k = v_{k+1}$, for $k \geq 0$.*
(b) *The subspace $L \subset V$ spanned by the vectors $\{v_k \mid k \geq 0\}$ is an $\mathfrak{sl}(2)$-submodule and all nonzero vectors v_k are linearly independent.*
(c) *Suppose that $v_k = 0$ for some k. Then there exists $\ell \in \mathbb{Z}_{\geq 0}$ such that $\lambda = \ell$, $v_k \neq 0$ for $0 \leq kle\ell$ and $v_k = 0$ for all $k > \ell$.*

Proof. Part (a) follows by induction on k, using the commutation relations between e, h, and f. Part (b) follows from (a) since the eigenvectors v_k have distinct eigenvalues. For (c), let ℓ be the first index such that $v_{\ell+1} = 0$. Then $0 = e \cdot v_{\ell+1} = (\ell + 1)(\lambda - \ell)v_\ell$, so $\lambda = \ell$.

□

2.2. **Irreducible finite dimensional $\mathfrak{sl}(2)$-modules.** For every $\ell \geq 0$, we construct an irreducible representation $V(\ell)$ of dimension $\ell + 1$ generated by a highest weight vector of weight ℓ.

Algebraic construction. The relations in Lemma 2.6 tell us how to define the module $V(\ell)$. Let $V(\ell)$ be the span of $\{v_0, v_1, \dots, v_\ell\}$ and define the $\mathfrak{sl}(2)$-action by:

$$h \cdot v_k = (\ell - 2k)v_k,\ e \cdot v_0 = 0,\ e \cdot v_{k+1} = (k+1)(l-k)v_k,\ f \cdot v_k = v_{k+1},\ k \geq 0. \tag{2.2.1}$$

(By convention, $v_{\ell+1} = 0$ in the above equations.)

From Lemma 2.6, we know that the actions of h, e, f are compatible with the relations between these elements, and hence, define an action of $\mathfrak{sl}(2)$ indeed.

Lemma 2.7. *The module $V(\ell)$ just defined is irreducible.*

Proof. Suppose that $M \neq 0$ is a submodule of $\mathfrak{sl}(2)$. Let $0 \neq \sum_{i=0}^{\ell} a_i v_i$ be a vector in M. Apply f to it: $f \cdot \sum_{i=0}^{\ell} a_i v_i = \sum_{i=1}^{\ell} a_{i-1} v_i$, which has to be an element of M too. Applying f repeatedly, we get that $a_0 v_\ell$ belongs to M and so $v_\ell \in M$. Then also $\sum_{i=0}^{\ell-1} a_i v_i$ is in M and repeat the process to show that all v_i are in M. So $M = V(\ell)$. □

Geometric construction.[2] Consider the action of $\mathfrak{g} = \mathfrak{sl}(2)$ on polynomials in two variables x and y via the operators

$$e \mapsto x\partial_y, \quad h \mapsto x\partial_x - y\partial_y, \quad f \mapsto y\partial_x. \tag{2.2.2}$$

One may verify directly that these assignments respect the $\mathfrak{sl}(2)$ relations. It is clear that these three operators preserve the total degree of any monomial. Therefore, the subspace $V(\ell)$ of homogeneous polynomials of degree ℓ is invariant under this action.

Notice that $V(\ell)$ is the span of $\{x^\ell, x^{\ell-1}y, x^{\ell-2}y^2, \dots, y^\ell\}$, so it is $\ell + 1$ dimensional. It is easy to verify that

$$e \cdot x^\ell = 0, \quad h \cdot (x^{\ell-i}y^i) = (\ell - 2i)x^{\ell-i}y^i, \quad f \cdot (x^{\ell-i}y^i) = (\ell - i)x^{\ell-i-1}y^{i+1},$$

so that the correspondence

$$x^{\ell-i}y^i \longleftrightarrow (\ell - i)!v_i$$

defines an isomorphism between this realization of the module of $V(\ell)$ and the algebraic one defined before.

Theorem 2.8. (1) *Every finite dimensional (nonzero) $\mathfrak{sl}(2)$-module V contains a submodule isomorphic to one of the $V(\ell)$'s.*
(2) *The Casimir element C acts on $V(\ell)$ by $\ell(\ell + 2)$.*
(3) *The modules $V(\ell)$ are irreducible, distinct, and exhaust all (isomorphism classes of) finite dimensional irreducible $\mathfrak{sl}(2)$-modules.*

Proof. (1) Consider all eigenvalues of V with respect to the action of h. Since V is finite dimensional, there exists an eigenvalue λ such that $\lambda + 2$ is not an eigenvalue. Let $v_0 \neq 0$ be an eigenvector for this λ. By Lemma 2.6, $\lambda = \ell$ for some ℓ and $L = V(\ell) \subset V$.

(2) We compute directly that $C \cdot v_0 = \ell(\ell+2)v_0$. If v is some other vector in $V(\ell)$, there exists $x \in U(\mathfrak{sl}(2))$ such that $v = x \cdot v_0$. Then $C \cdot v = Cx \cdot v_0 = xC \cdot v_0 = \ell(\ell + 2)x \cdot v_0 = \ell(\ell + 2)v$.

(3) We have already proved that the modules $V(\ell)$ are irreducible. Since the scalar by which C acts on each $V(\ell)$ determines ℓ uniquely, it follows that these modules are non-isomorphic.

□

Remark 2.9. *From the construction of the modules $V(\ell)$, we see that on every $V(\ell)$, the element h acts semisimply and the weights are $\{\ell, \ell - 2, \ell - 4, \dots, -\ell + 2, -\ell\}$ and each weight space is one dimensional.*

[2]The reason we refer to this construction as geometric is the following. Consider the group $G = SL(2, \mathbb{C})$ of 2×2 matrices of determinant one acting via matrix multiplication on the space $\mathbb{C}^2 = \{(x, y)\}$. There is an induced action on polynomials in x and y and the action of the Lie algebra $\mathfrak{g}$ defined ad-hoc in this paragraph is in fact the differential (in the Lie groups sense) of the natural action of G on polynomials.

2.3. **Complete reducibility.** In this subsection, we prove directly that every finite dimensional $\mathfrak{sl}(2)$-module is completely reducible. This completes the classification of finite dimensional $sl(2,\mathbb{C})$-modules.

Proposition 2.10. *Every finite dimensional $\mathfrak{sl}(2)$-module V is isomorphic to a direct sum of modules $V(\ell)$, $\ell \geq 0$. In particular, V is completely reducible.*

Proof. (Bernstein) We'll use a general criterion whose proof is an exercise: if every module of length 2 is completely reducible, then every module of finite length is completely reducible. This reduces the proof to the case when V has length 2 with simple submodule $S = V(\ell)$ and simple quotient $Q = V/S \cong V(k)$.

If $k \neq \ell$, then the Casimir element acts with different eigenvalues on S and Q. Therefore, V splits into a direct sum of two generalized eigenspaces for C, one with eigenvalue $\ell(\ell+1)$ and the other with eigenvalue $k(k+1)$. Since C is central in $U(\mathfrak{sl}(2))$, both of these eigenspaces are $\mathfrak{sl}(2)$-submodules and we are done.

Assume $k = \ell$. Decompose V into generalized h-eigenspaces $V = \oplus V_i$. By assumption, $i \in \{-\ell, \ell+2, \dots, \ell-2, \ell\}$ and $\dim V_i = 2$. We claim that $f^\ell : V_\ell \to V_{-\ell}$ is a linear isomorphism. Let $0 \neq v \in V_\ell$ be given. If $v \in S$, then v is a highest weight vector with weight ℓ and so $f^\ell v \neq 0$. Otherwise, $v + S \neq S$ in $Q = V/S$, but $v + S \in Q_\ell$, so $f^\ell(v+S) \neq S$, implying that $f^\ell v \notin S$.

Now consider the identity

$$ef^{\ell+1} - f^{\ell+1}e = f^\ell(h-\ell)$$

acting on V_ℓ. Since the left hand side is 0, the right hand side must be 0 too. But f^ℓ is invertible on V_ℓ as we argued before, which means that $h = \ell \cdot \text{ Id}$ on V_ℓ. In other words, $V_\ell = V_\ell^{ss}$. But this gives two linearly independent highest weight vectors with weight ℓ in V, and V decomposes as the sum of the $\mathfrak{sl}(2)$-submodules that these two vectors generate.

□

Corollary 2.11. *Let V be a finite dimensional $\mathfrak{sl}(2)$-module. Then h acts semisimply on V and for every weight $i \geq 0$, $f^i : V_i \to V_i$ and $e^i : V_{-i} \to V_i$ are linear isomorphisms.*

Proof. Both claims follow from the complete reducibility of V and the corresponding statements (which we know are true) for the modules $V(\ell)$. □

3. Some basic facts about semisimple Lie algebras

In this section, we recall a few basic definitions and results about the structure of semisimple Lie algebras. These results will be used in the sequel. We do not give proofs of these facts, but a complete treatment (including proofs) can be found in Kevin McGerty's notes for the Michaelmas course in Lie algebras C2.1.

In this section, $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{C}$.

3.1. **Nilpotent and solvable Lie algebras.** The *lower central series* of $\mathfrak{g}$ is the decreasing chain of ideals $C^0\mathfrak{g} \supseteq C^1\mathfrak{g} \supseteq C^2\mathfrak{g} \supseteq \dots \supseteq C^i\mathfrak{g} \supseteq \dots$ defined inductively by $C^0\mathfrak{g} = \mathfrak{g}$ and $C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}]$ for $i \geq 1$. We say that $\mathfrak{g}$ is nilpotent if there exists $N > 0$ such that $C^N\mathfrak{g} = 0$.

The *derived series* of $\mathfrak{g}$ is the decreasing chain of ideals $D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq D^2\mathfrak{g} \supseteq \dots \supseteq D^i\mathfrak{g} \supseteq \dots$ defined inductively by $D^0\mathfrak{g} = \mathfrak{g}$ and $D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}]$ for $i \geq 1$. We say that $\mathfrak{g}$ is solvable if there exists $N > 0$ such that $D^N\mathfrak{g} = 0$. The ideal $D^1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ is called the derived subalgebra of $\mathfrak{g}$, and it is also denoted by $D\mathfrak{g}$.

Since for every i, $D^i\mathfrak{g} \subseteq C^i\mathfrak{g}$, it is clear that every nilpotent Lie algebra is solvable. The converse is false, as we may see from the following example.

Example 3.1. *Let V be a vector space of dimension n. A complete flag $\mathcal{F}$ in V is a collection of vector subspaces $\mathcal{F} = (V_0 \subset V_1 \subset \dots \subset V_n)$, where $\dim V_i = i$. Define*

$$\mathfrak{n}_\mathcal{F} = \{x \in \mathfrak{g}l(V) \mid x(V_i) \subset V_{i-1}\}, \quad \mathfrak{b}_\mathcal{F} = \{x \in \mathfrak{g}l(V) \mid x(V_i) \subset V_i\}. \tag{3.1.1}$$

The algebra $\mathfrak{b}_\mathcal{F}$ is called the stabilizer of the flag $\mathcal{F}$. Then one can show that $\mathfrak{b}_\mathcal{F}$ is a solvable Lie algebra (but not nilpotent), $\mathfrak{n}_\mathcal{F}$ is nilpotent, and $\mathfrak{n}_\mathcal{F}$ is the derived subalgebra of $\mathfrak{b}_\mathcal{F}$.

If we choose a basis $\{e_1, \dots, e_n\}$ of V and set $V_i = span\{e_1, \dots, e_i\}$, then $\mathfrak{b}_\mathcal{F}$ is identified with the algebra of upper triangular matrices with respect to this basis, while $\mathfrak{n}_\mathcal{F}$ is the algebra of strictly upper triangular matrices.

It is easy to see that if I and J are two solvable ideals of $\mathfrak{g}$, then $I + J$ is also a solvable ideal. This implies that there exists a unique maximal solvable ideal in $\mathfrak{g}$, called the radical of $\mathfrak{g}$, $\text{rad}(\mathfrak{g})$.

Definition 3.2. *A Lie algebra $\mathfrak{g}$ is called semisimple if rad$(\mathfrak{g}) = 0$. Recall that a Lie algebra $\mathfrak{g}$ is called simple if $\mathfrak{g}$ doesn't have any proper ideals.*[3]

3.2. **The Killing form.** The Killing form of $\mathfrak{g}$ is the pairing

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad \kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)),\ x, y \in \mathfrak{g}. \tag{3.2.1}$$

It is a symmetric, bilinear form on $\mathfrak{g}$ and it is $\mathfrak{g}$-invariant, meaning that

$$\kappa([x, y], z) + \kappa(y, [x, z]) = 0, \quad x, y, z \in \mathfrak{g}. \tag{3.2.2}$$

An important criterion is the following:

Theorem 3.3 (Cartan's criterion for semisimplicity)**.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form κ is nondegenerate.*

Using this criterion, one can prove that $\mathfrak{g}$ is semisimple if and only if it is a direct sum of simple ideals. (Moreover, the decomposition into a sum of simple ideals is unique.)

Example 3.4. *The classical Lie algebras $\mathfrak{sl}(V)$, $\mathfrak{so}(V)$ (defined with respect to a nondegenerate symmetric bilinear form), and $\mathfrak{sp}(V)$ (with respect to a nondegenerate skew-symmetric bilinear form) are all simple, hence semisimple, Lie algebras.*

3.3. **Cartan subalgebras.** A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a *Cartan subalgebra* if it has the following properties:

(1) $\mathfrak{h}$ is nilpotent;
(2) $\mathfrak{h}$ is self-normalizing, i.e.,

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid \mathrm{ad}(x)(\mathfrak{h}) \subset \mathfrak{h},\ \text{for all } x \in \mathfrak{g}\}.$$

Cartan subalgebras exist. To construct them, consider subalgebras $\mathfrak{g}_{0,x} = \{y \in \mathfrak{g} \mid \mathrm{ad}(x)^N y = 0,\ \text{for some } N > 0\}$, in other words, the generalized eigenspaces of $\mathrm{ad}(x)$, for some $x \in \mathfrak{g}$. An element x is called regular if $\dim \mathfrak{g}_{0,x}$ is minimal among all such subalgebras. One can show that every subalgebra $\mathfrak{g}_{0,x}$, where x is regular, is a Cartan subalgebra.

Proposition 3.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra. The following are equivalent:*

(a) *$\mathfrak{h}$ is a Cartan subalgebra;*
(b) *$\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ which is toral, i.e., the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is semisimple.*

Moreover, any two Cartan subalgebras are conjugate (under the adjoint action).

Example 3.6. *If $\mathfrak{g} = \mathfrak{s}l(n)$, then the usual choice of Cartan subalgebra is $\mathfrak{h}$ consisting of diagonal, trace 0, matrices.*

3.4. **Cartan decomposition.** From now on, $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h}$ is a fixed Cartan subalgebra. The main tool for the structure of $\mathfrak{g}$ is the Cartan decomposition.

Decompose $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian, basic linear algebra tells us that $\mathfrak{g}$ decomposes into a direct sum of generalized $\mathfrak{h}$-eigenspaces:

$$\mathfrak{g} = \bigoplus_{\chi \in \mathfrak{h}^*} \mathfrak{g}_\chi,\ \mathfrak{g}_\chi = \{x \in \mathfrak{g} \mid \text{for all } h \in \mathfrak{h} \text{ there exists } N > 0 \text{ such that } (\mathrm{ad}(h) - \chi(h))^N x = 0\}.$$

Since $\mathfrak{h}$ acts semisimply, the generalized eigenspaces are just the usual eigenspaces: $\mathfrak{g}_\chi = \{x \in \mathfrak{g} \mid [h, x] = \chi(h)x,\ h \in \mathfrak{h}\}$.

One can show first that $\mathfrak{g}_0 = \mathfrak{h}$. Denote $\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ and call it the set of *roots* of $\mathfrak{g}$ (with respect to $\mathfrak{h}$). The Cartan (or root) decomposition of $\mathfrak{g}$ is then:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{3.4.1}$$

The spaces $\mathfrak{g}_\alpha$ are called root spaces and every nonzero vector in $\mathfrak{g}_\alpha$ is called a root vector. Here is a list of the main facts about this decomposition. The main tool for proving the nontrivial statements is the Cartan criterion for semisimplicity.

[3]Because of the skew-symmetry of the Lie bracket in $\mathfrak{g}$, left ideals are the same as right ideals and are the same as two-sided ideals.

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, $\alpha, \beta \in \Phi \cup \{0\}$.
(2) $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$. Moreover, κ gives a nondegenerate pairing of $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$ and the restriction of κ to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate. The Killing form on $\mathfrak{h}$ can be computed by the formula

$$\kappa(h, h') = \sum_{\alpha\in\Phi} \alpha(h)\alpha(h'), \quad h, h' \in \mathfrak{h}. \tag{3.4.2}$$

(3) Φ spans $\mathfrak{h}^*$.
(4) $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.
(5) For every $\alpha \in \Phi$, there exist vectors $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_\alpha$, $h_\alpha = [e_\alpha, e_{-\alpha}] \in \mathfrak{h}$ such that $\{e_\alpha, h_\alpha, e_{-\alpha}\}$ satisfy the $\mathfrak{sl}(2)$-relations. The element h_α is called the coroot corresponding to α and the content of the claim is that $\alpha(h_\alpha) = 2$. The coroots $\{h_\alpha \mid \alpha \in \Phi\}$ span $\mathfrak{h}$.
(6) If $\alpha + \beta \notin \Phi$ then $[e_\alpha, e_\beta] = 0$ (obviously). If $\alpha + \beta \in \Phi$, then $[e_\alpha, e_\beta] = Ce_{\alpha+\beta}$ for some nonzero scalar C.

Remark 3.7. *If we regard $\mathfrak{g}$ as an $\mathfrak{sl}(2)$-module via the adjoint action, for the $\mathfrak{sl}(2)$ spanned by $\{e_\alpha, h_\alpha, e_{-\alpha}\}$, then we know that the weights of this representation must be integers. But every $\beta(h_\alpha)$, $\beta \in \Phi$ is a weight, thus $\beta(h_\alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.*

Definition 3.8. *Define:*

$$\begin{aligned} Q^\vee &= \{\sum_{\alpha\in\Phi} a_\alpha h_\alpha \mid a_\alpha \in \mathbb{Z}\} \subset \mathfrak{h}; \\ Q &= \{\sum_{\alpha\in\Phi} a_\alpha \alpha \mid a_\alpha \in \mathbb{Z}\} \subset \mathfrak{h}^*; \\ P &= \{\chi \in \mathfrak{h}^* \mid \chi(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}. \end{aligned} \tag{3.4.3}$$

These groups are called the coroot, root, weight lattice, respectively. In light of the previous remark, we have $Q \subseteq P$, but in general, they are not equal.

Example 3.9. *Let $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{h}$ be the diagonal matrices of trace 0. In coordinates, we may think of $\mathfrak{h}$ as $\mathfrak{h} = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid \sum_i a_i = 0\}$. The dual space $\mathfrak{h}^*$ is naturally identified with $\mathfrak{h}^* = \mathbb{C}\langle \epsilon_1, \dots, \epsilon_n\rangle / \langle \epsilon_1 + \epsilon_2 + \dots + \epsilon_n\rangle$, where $\epsilon_i : \mathfrak{h} \to \mathbb{C}$ is defined by $\epsilon_i(a_1, a_2, \dots, a_n) = a_i$.*

The roots are $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n\} \subset \mathfrak{h}^*$. The Cartan decomposition is $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i\ne j} \mathfrak{g}_{\epsilon_i - \epsilon_j}$, with $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C} \cdot E_{ij}$, where E_{ij} is the elementary matrix that has 1 on the (i, j) position and 0 everywhere else.

The lattices in the previous definition are:

$$\begin{aligned} Q^\vee &= \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum_i a_i = 0\}; \\ Q &= \{\sum_{i=1}^n a_i \epsilon_i \mid a_i \in \mathbb{Z},\ \sum_i a_i = 0\} / \langle \epsilon_1 + \epsilon_2 + \dots + \epsilon_n\rangle; \\ P &= \mathbb{Z}\langle \epsilon_1, \dots, \epsilon_n\rangle / \langle \epsilon_1 + \epsilon_2 + \dots + \epsilon_n\rangle. \end{aligned} \tag{3.4.4}$$

In this case, $Q \subsetneq P$ and, in fact, one may show that $P/Q \cong \mathbb{Z}/n\mathbb{Z}$.

The Killing form is nondegenerate on $\mathfrak{h}$, and therefore it induces a nondegenerate bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{h}^*$. The coroot h_α is uniquely determined by the property

$$\chi(h_\alpha) = \frac{2\langle \chi, \alpha\rangle}{\langle \alpha, \alpha\rangle}, \text{ for all } \chi \in \mathfrak{h}^*.$$

Moreover, since $\kappa(h, h) = \sum_{\alpha\in\Phi} \alpha(h)^2$, it follows that κ is positive definite on $Q^\vee$. Dually, $\langle\ ,\ \rangle$ is positive definite on P.

3.5. **The Weyl group.** For every $\alpha \in \Phi$, define the reflection in the hyperplane perpendicular to α:

$$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*, \quad s_\alpha(\chi) = \chi - \chi(h_\alpha)\alpha. \tag{3.5.1}$$

By the usual abuse of notation, we also denote by s_α the corresponding reflection in $\mathfrak{h}$:

$$s_\alpha : \mathfrak{h} \to \mathfrak{h}, \quad s_\alpha(h) = h - \alpha(h)h_\alpha. \tag{3.5.2}$$

Then $s_\alpha^2 = \mathrm{Id}$ and $\det(s_\alpha) = -1$. The Weyl group W is the subgroup of $GL(\mathfrak{h}^*)$ (respectively, $GL(\mathfrak{h})$) generated by s_α. In fact, since all s_α's preserve the Killing form, W is a subgroup of the orthogonal group $O(\mathfrak{h}^*, \langle\ ,\ \rangle)$ (respectively, $O(\mathfrak{h}, \kappa)$).

In order to use Euclidean geometry, introduce the real vector spaces $\mathfrak{a} = \mathbb{R} \otimes_{\mathbb{Z}} Q^\vee \subset \mathfrak{h}$ and $\mathfrak{a}^* = \mathbb{R} \otimes_{\mathbb{Z}} Q$. These spaces are endowed with positive definite forms coming from κ on $\mathfrak{a}$, and $\langle\ ,\ \rangle$ on $\mathfrak{a}^*$.

If $\alpha \in \Phi$, denote the hyperplane perpendicular to α by $H_\alpha = \{\chi \in \mathfrak{a}^* \mid \chi(h_\alpha) = 0\}$.

Definition 3.10. *The connected components of $\mathfrak{a}^* \setminus \cup_{\alpha \in \Phi} H_\alpha$ are called Weyl chambers. Fix a Weyl chamber $\mathcal{C}$ and let $\overline{\mathcal{C}}$ denote its closure in $\mathfrak{a}^*$.*

Proposition 3.11. *$\overline{\mathcal{C}}$ is a fundamental domain for the action of W on $\mathfrak{a}^*$, meaning that:*

1. *if $\chi \in \mathfrak{a}^*$ then there exists $w \in W$ such that $w\chi \in \overline{\mathcal{C}}$;*
2. *if $\chi, w\chi \in \overline{\mathcal{C}}$ then $\chi = w\chi$.*

Moreover, if $\chi \in \mathcal{C}$ and $\chi = w\chi$, then $w = 1$.

From this result, it follows that W permutes the (open) Weyl chambers transitely and in particular, the number of Weyl chambers equals $|W|$.

We are now in position to define positive and simple roots (with respect to our fixed choice of Weyl chamber $\mathcal{C}$).

Definition 3.12. *A root $\alpha \in \Phi$ is called positive if $\langle \chi, \alpha \rangle > 0$ for all $\chi \in \mathcal{C}$. This means that α makes acute angles with every vector in $\mathcal{C}$. Denote by Φ^+ the subset of positive roots and let $\Phi^- = -\Phi^+$ be the negative roots. Of course, $\Phi = \Phi^+ \sqcup \Phi^-$. Notice that Φ^+ is closed under addition.*

A positive root α is called a simple root if it cannot be written as a sum of more than one positive roots. Let $\Pi \subset \Phi^+$ denote the set of simple roots.

The basic facts about positive and simple roots are the following:

1. Π is a $\mathbb{Z}$-basis of Q (resp., $\mathbb{R}$-basis of $\mathfrak{a}^*$, resp., $\mathbb{C}$-basis of $\mathfrak{h}^*$). Every positive root α can be written as a sum of simple roots with nonnegative integer coefficients.
2. A positive root α is simple if and only if H_α is a wall of the fundamental chamber $\mathcal{C}$.
3. W is generated by s_α, $\alpha \in \Pi$.
4. If α, β are simple root and $\alpha \neq \beta$ then $\langle \alpha, \beta \rangle \leq 0$.

We define a partial order on $\mathfrak{h}^*$ that we will use later. Let Q^+ denote the $\mathbb{Z}_{\geq 0}$-span of Π. Then we say that $\chi_1 \leq \chi_2$ if $\chi_2 - \chi_1 \in Q^+$.

A weight $\chi \in P$ is called *dominant* if $\chi(h_\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Pi$. This is equivalent with $s_\alpha(\chi) \leq \chi$ for all $\alpha \in \Pi$. Denote by P^+ the semigroup of dominant weights. Clearly, $Q^+ \subseteq P^+$.

Example 3.13. *Denote*

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in \mathfrak{h}^*. \tag{3.5.3}$$

One may show that $\rho(h_\alpha) = 1$ for all $\alpha \in \Pi$. In particular, $\rho \in P^+$.

Example 3.14. *In the $\mathfrak{sl}(n)$ example, the usual choice of positive roots is $\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\}$. The corresponding simple roots are $\Pi = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\}$. The Weyl group is the symmetric group S_n acting on coordinates in the standard way. The weight ρ equals in coordinates $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2})$.*

3.6. **Some more Weyl group combinatorics.** Let Π be a base of Φ. The proof of the following lemma is left as an exercise.

Lemma 3.15. *If $\alpha \in \Pi$, then the reflection s_α permutes the roots in $\Phi^+ \setminus \{\alpha\}$.*

We prove a basic result about Weyl group elements.

Lemma 3.16. *Suppose $\alpha_1, \alpha_2, \dots, \alpha_m$ are in Π, not necessarily distinct. Write $s_i = s_{\alpha_i}$. If*

$$s_1 \cdot \ldots \cdot s_{m-1}(\alpha_m) \in \Phi^-$$

then for some index $1 \leq k < m$, $s_1 \cdot \ldots \cdot s_m = s_1 \cdot \ldots \cdot s_{k-1} s_{k+1} \cdot \ldots \cdot s_{m-1}$.

Proof. Write $\beta_i = s_{i+1} \cdot \ldots \cdot s_{m-1}(\alpha_m)$, $0 \le i \le m-2$ and $\beta_{m-1} = \alpha_m$. Then $\beta_0 \in \Phi^-$ and $\beta_{m-1} \in \Phi^+$. Let k be the smallest index such that $\beta_k \in \Phi^+$. Then $\beta_{k-1} = s_k(\beta_k) \in \Phi^-$. But s_k permutes all positive roots except α_k, hence $\beta_k = \alpha_k$. So $\alpha_k = s_{k+1} \cdot \ldots \cdot s_{m-1}(\alpha_m$ and therefore $s_k = (s_{k+1} \cdot \ldots \cdot s_{m-1})s_m(s_{m-1} \cdot \ldots \cdot s_{k+1})$, from which the lemma follows. □

Since W is generated by $\{s_\alpha \mid \alpha \in \Pi\}$, we may write each w as a product of simple reflections. If $w = s_{\alpha_1} \cdots s_{\alpha_m}$, $\alpha_i \in \Pi$, such that m is minimal, then we say that this is a *reduced expression* for w, and we call m the *length* of w. Write $\ell(w) = m$. Clearly, $\ell(1) = 0$, $\ell(s_\alpha) = 1$ for $\alpha \in \Pi$, and $\ell(w) = \ell(w^{-1})$.

Lemma 3.17. *For all $w \in W$, $\ell(w) = \#\{\beta \in \Phi^+ \mid w(\beta) < 0\}$.*

Proof. The proof is by induction of $\ell(w)$, the base case $w = 1$ being clear. Denote $n(w) = \#\{\beta \in \Phi^+ \mid w(\beta) < 0\}$. Suppose $w = s_{\alpha_1} \cdots s_{\alpha_m}$ is a reduced expression. Set $\alpha = \alpha_m$. Then $w(\alpha) < 0$. On the other hand, $n(ws_\alpha) = n(w) - 1$, because s_α permutes all positive roots except α. Also $\ell(ws_\alpha) = \ell(w) - 1 < \ell(w)$, so by induction $\ell(ws_\alpha) = n(ws_\alpha)$, hence $\ell(w) = n(w)$. □

Proposition 3.18. *Suppose $\lambda, \mu \in \overline{\mathcal{C}}$. If $w\lambda = \mu$ for some $w \in W$ then w is a product of simple reflections each of which fixes λ. In particular, $\lambda = \mu$.*

Proof. We prove the claim by induction on $\ell(w)$. If $\ell(w) > 0$, there exists $\beta \in \Phi^+$ such that $w(\beta) \in \Phi^-$, so there exists a simple root α such that $w(\alpha) \in \Phi^-$. Then

$$0 \ge \langle \mu, w(\alpha) \rangle = \langle w^{-1}(\mu), \alpha \rangle = \langle \lambda, \alpha \rangle \ge 0,$$

hence $\langle \lambda, \alpha \rangle = 0$. This means that $s_\alpha(\lambda) = \lambda$ and $(ws_\alpha)(\lambda) = \mu$. Since $\ell(ws_\alpha) = \ell(w) - 1$, we can continue by induction. □

4. The category $\mathcal{O}$

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}$ and let Φ be the roots of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the Cartan decomposition. Fix a choice of positive roots Φ^+ and let Π be the corresponding simple roots. We retain all the other notation from the previous section. Denote

$$\mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+. \tag{4.0.1}$$

It is easy to prove the following lemma by using the commutation relations between $\mathfrak{h}$, e_α, $\alpha \in \Phi^+$.

Lemma 4.1. *The subalgebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent. The subalgebra $\mathfrak{b}$ is solvable and its derived subalgebra is $\mathfrak{n}^+$.*

As a consequence of the PBW theorem, we have the following *triangular decomposition* of $U(\mathfrak{g})$:

$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+). \tag{4.0.2}$$

We will use this decomposition repeatedly in this section.

4.1. **Definitions.** Let V be an $\mathfrak{g}$-module. For $\lambda \in \mathfrak{g}^*$, denote by V_λ the generalized λ-weight space and by V_λ^{ss} the λ-weight space. Recall that:

$$V_\lambda = \{v \in V \mid \text{ for every } h \in \mathfrak{h} \text{ there is } N > 0 \text{ such that } (h - \lambda(h))^N v = 0\}.$$

The same argument as in $\mathfrak{sl}(2)$ shows that

$$e_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}, \quad e_\alpha \cdot V_\lambda^{ss} \subseteq V_{\lambda+\alpha}^{ss}, \quad \alpha \in \Phi. \tag{4.1.1}$$

We say that V is $\mathfrak{h}$-*semisimple* if $V_\lambda = V_\lambda^{ss}$ for all $\lambda \in \mathfrak{h}^*$ and $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. The set

$$\Psi(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \ne 0\} \tag{4.1.2}$$

is called the set of *weights* of V.

Definition 4.2. *The category $\mathcal{O}$ of $\mathfrak{g}$ is the full subcategory of (left) $U(\mathfrak{g})$-modules whose objects M satisfy the following conditions:*

- ($\mathcal{O}1$) *M is a finitely generated $U(\mathfrak{g})$-module;*
- ($\mathcal{O}2$) *M is $\mathfrak{h}$-semisimple;*
- ($\mathcal{O}3$) *M is locally $\mathfrak{n}^+$-finite, i.e., for every $v \in M$, the subspace $U(\mathfrak{n}^+) \cdot v$ is finite dimensional.*

Example 4.3. *Recall the modules constructed for* $\mathfrak{sl}(2)$, $M(\lambda) = span\{v_0, v_1, \dots, v_n, \dots\}$. *Given the explicit construction, we can see immediately that these modules are in the category* $\mathcal{O}$.

Lemma 4.4. *Every finite dimensional* $\mathfrak{g}$*-module is in* $\mathcal{O}$.

Proof. If M is finite dimensional, then ($\mathcal{O}1$) and ($\mathcal{O}3$) are automatic. For ($\mathcal{O}2$), recall that $\mathfrak{h}$ is the span of the coroots $\{h_\alpha \mid \alpha \in \Phi\}$. Since h_α commute, a standard linear algebra fact implies that the action of $\mathfrak{h}$ is semisimple (diagonalizable) if and only if every h_α is semisimple. So it is sufficient to prove that the action of h_α for a single $\alpha \in \Phi$ is semisimple. Regard h_α as part of the Lie triple $\{e_\alpha, h_\alpha, e_{-\alpha}\}$ which spans an $\mathfrak{sl}(2)$. Regarding V as a finite dimensional $\mathfrak{sl}(2)$-module, the results from $\mathfrak{sl}(2)$ tell us that V is h_α-semisimple. □

Recall the semigroup Q^+ and the partial order $<$.

Lemma 4.5. *Let* M *be a module in* $\mathcal{O}$.

1. *For every* $\lambda \in \Psi(M)$, *the weight space* M_λ *is finite dimensional.*
2. *There exist finitely many weights* $\lambda_1, \dots, \lambda_k \in \Psi(M)$ *such that for every* $\lambda \in \Psi(M)$, $\lambda < \lambda_i$ *for some* i.

Proof. Using ($\mathcal{O}2$) we can take a generating set of M to consist of weight vectors. To prove (1) and (2), it then suffices to consider the case when $M = U(\mathfrak{g}) \cdot v_\lambda$, where v_λ is a λ-weight vector. We'll use the triangular decomposition of $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})(U(\mathfrak{n}^+)$. Notice first that $V := U(\mathfrak{n}^+) \cdot v_\lambda$ is finite dimensional ($\mathcal{O}3$) and spanned by weight vectors for weights in $\lambda + Q^+$. Let $\{v_{\lambda_1}, \dots, v_{\lambda_k}\}$ be a basis of V consisting of weight vectors. Next V is stable under the action of $U(\mathfrak{h})$. The action of $U(\mathfrak{n}^-)$ produces weights lower than $\lambda_1, \dots, \lambda_k$. More precisely, the weight vectors are all of the form $e_{-\alpha_1}^{i_1} \cdots e_{-\alpha_\ell}^{i_\ell} \cdot v_{\lambda_i}$ with corresponding weight $\mu = \lambda_i - \sum_{j=1}^m i_j \alpha_j$. Here all i_j are nonnegative integers.

Finally, to see that all weight spaces are finite dimensional, notice that there are only finitely many ways to write μ in the above form. □

4.2. **Basic properties.** We record some of the immediate properties of $\mathcal{O}$ in the next proposition.

Proposition 4.6.
1. $\mathcal{O}$ *is a Noetherian category, i.e., every* $M \in \mathcal{O}$ *is a Noetherian* $U(\mathfrak{g})$*-module.*
2. $\mathcal{O}$ *is closed under taking submodules, quotients, and finite direct sums. Hence* $\mathcal{O}$ *is an abelian category.*
3. *If* $M \in \mathcal{O}$ *and* L *is finite dimensional, then* $L \otimes M \in \mathcal{O}$.
4. *If* $M \in \mathcal{O}$, *them* M *is* $Z(\mathfrak{g})$*-finite, i.e., for every* $v \in M$, *span*$\{z \cdot v \mid z \in Z(\mathfrak{g})\}$ *is finite dimensional.*
5. *If* $M \in \mathcal{O}$, *then* M *is finitely generated as a* $U(\mathfrak{n}^-)$*-module.*

Proof. (1) $U(\mathfrak{g})$ is Noetherian[4] and M is a finitely generated $U(\mathfrak{g})$-module. Therefore M is Noetherian.

(2) The only statement that needs explanation is the fact that ($\mathcal{O}1$) holds for submodules. But this is precisely because of the Noetherian property from (1): every submodule of a finitely generated module is finitely generated.

(3) The tensor product $L \otimes M$ satisfies ($\mathcal{O}2$) and ($\mathcal{O}3$). To prove finite generation, let $\{v_1, \dots, v_n\}$ be a basis of L and let $m_1, \dots, m_k$ generate M. Then $\{v_i \otimes m_j\}$ generates $L \otimes M$. To see this, let N be the submodule this set generates. Since every $v \in L$ can be written as $v = \sum a_i v_i$, we see that all simple tensors of the form $v \otimes m_j$, $v \in L$, are also in N. If $x \in \mathfrak{g}$, we calculate

$$x \cdot (v \otimes m_j) = x \cdot v \otimes m_j + v \otimes x \cdot m_j \in N.$$

The first term is in N, so $v \otimes x \cdot m_j \in N$. Repeating this, we see that $v \otimes u \cdot m_j \in N$ for all $u \in U(\mathfrak{g})$. But then $L \otimes M \subset N$, which concludes the proof.

(4) If $v \in$M we may write v as a sum of weight vectors. It is sufficient to prove the claim when $v \in M_\lambda$. Since $z \in Z(\mathfrak{g})$ commutes with $\mathfrak{h}$, we see that $z \cdot v \in M_\lambda$ as well. But M_λ is finite dimensional, so $Z(\mathfrak{g}) \cdot v$ is finite dimensional.

(5) Because of the axioms, we see that M is generated by a finite dimensional $U(\mathfrak{b})$-module V. By the PBW theorem, a basis of V generates M as a $U(\mathfrak{n}^-)$-module. □

Example 4.7. *We can verify that when* $\mathfrak{g} = \mathfrak{sl}(2)$, *the tensor product* $M(\lambda) \otimes M(\mu)$ *is not in* $\mathcal{O}$.

[4] This is proved in the "Noncommutative rings" lectures using the PBW theorem, so we won't repeat the proof here.

4.3. **Highest weight modules.**

Definition 4.8. *Let M be a $U(\mathfrak{g})$-module. A nonzero vector $v \in M$ is called a highest weight vector (of weight λ) if $v \in M_\lambda$ for some $\lambda \in \Psi(M)$ and $\mathfrak{n}^+ \cdot v = 0$. The last condition is equivalent with $e_\alpha \cdot v = 0$ for all $\alpha \in \Phi^+$.*

From Lemma 4.5, every $M \in \mathcal{O}$ has a highest weight vector. (Pick λ a maximal weight with respect to $<$.) Therefore, it makes sense to make the following definition.

Definition 4.9. *A $U(\mathfrak{g})$-module M is called a highest weight module of weight λ if there exists $v \in M_\lambda$ such that $M = U(\mathfrak{g}) \cdot v$. The last condition is equivalent with $M = U(\mathfrak{n}^-) \cdot v$ by the triangular decomposition.*

We list several immediate properties of highest weight modules.

(a) Choose an ordering of the positive roots: $\alpha_1, \alpha_2, \dots, \alpha_m$. Then M is spanned by the vectors $e_{-\alpha_1}^{i_1} \cdots e_{-\alpha_m}^{i_m} \cdot v$. Each such vector has weight $\lambda - \sum_{j=1}^m i_j \alpha_j$. Hence M is $\mathfrak{h}$-semisimple.
(b) Arguing as in the proof of Proposition 4.6(5), we see that if $\mu \in \Psi(M)$, then $\mu \le \lambda$. Moreover, each M_μ is finite dimensional and, in particular, $\dim M_\lambda = 1$. Therefore $M \in \mathcal{O}$. (Axiom ($\mathcal{O}3$) follows from the fact that e_α, $\alpha \in \Phi^+$ maps M_μ to $M_{\mu+\alpha}$.)
(c) Each nonzero quotient of M is also a highest weight module of weight λ.

Proposition 4.10. *Let $M \in \mathcal{O}$ be a highest weight module.*

(1) *M has a unique maximal submodule and hence a unique simple quotient. In particular, M is indecomposable.*
(2) *Let $\lambda \in \mathfrak{h}^*$ be given. All simple highest weight modules of weight λ are isomorphic. If M is a simple highest weight module of weight λ, then $\dim \operatorname{End}_{U(\mathfrak{g})}(M) = 1$.*

Proof. (1) If N is a proper submodule of M then $N \in \mathcal{O}$ (as $M \in \mathcal{O}$), hence N is $\mathfrak{h}$-semisimple. Write $N = \bigoplus_{\mu \in \Psi(N) \subset \Psi(M)} N_\nu$. Since M_λ is one-dimensional and every vector in M_λ generated M, it follows that $\lambda \notin \Psi(N)$. This implies that the sum of all proper submodules of M is still proper (λ is not a weight for any of them), and therefore there is a unique maximal submodule.

(2) Suppose M_1 and M_2 are two simple highest weight modules of the same weight λ. Let v_1, v_2 be highest weight vectors for M_1 and M_2, respectively. Then $v = v_1 + v_2$ is also a highest weight vector in $M = M_1 \oplus M_2$. Denote $N = U(\mathfrak{g}) \cdot v \subset M$. Then N is a highest weight module of weight λ. The two canonical projections give projections $N \to M_1$ and $N \to M_2$. Hence M_1 and M_2 are both simple quotients of N. By (1), $M_1 \cong M_2$.

For the second part, let M be a simple highest weight module of weight λ and let $\phi : M \to M$ be a nonzero $\mathfrak{g}$-homomorphism. Since M is simple, ϕ must be an isomorphism. Then it maps M_λ to M_λ. Fix a highest weight vector $v \in M_\lambda$. Since M_λ is one-dimensional, $\phi(v) = cv$ for some constant $c \in \mathbb{C}$. But since v generates M, it follows that $\phi = c \cdot \mathrm{Id}$. □

As a consequence, we can see that the highest weight modules are the building blocks of category $\mathcal{O}$.

Corollary 4.11. *Let $M \neq 0$ be a module in $\mathcal{O}$. There exists a finite filtration*

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

of modules in $\mathcal{O}$ such that M_i/M_{i+1} is a highest weight module.

Proof. M is generated by finitely many weight vectors $v_{\lambda_1}, \dots, v_{\lambda_\ell}$. Set $V = U(\mathfrak{n}^+) \cdot \langle v_{\lambda_1}, \dots, v_{\lambda_\ell} \rangle$. Because of ($\mathcal{O}3$), V is finite dimensional and, of course, $M = U(\mathfrak{g}) \cdot V = U(\mathfrak{n}^-) \cdot V$. We proceed by induction on V.

If $\dim V = 1$, then M is a highest weight module itself. Otherwise, take $v \in V$ a weight vector for a maximal weight (among the weights that occur in V). Then v must be a highest weight vector in M. Set $M_1 = U(\mathfrak{n}^-) \cdot v$ which is a submodule of M and a highest weight module. Next $\bar{M} = M/M_1$ is generated by $\bar{V} = V/\langle v \rangle$. Since $\dim \bar{V} < \dim V$, we can finish the proof by induction. □

4.4. **Verma modules.** Recall the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Since $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}$, we have the natural projection $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n}^+ \cong \mathfrak{h}$, which is a Lie algebra homomorphism. For every $\lambda \in fh^*$, denote by $\mathbb{C}_\lambda$ the one-dimensional $\mathfrak{b}$-representation pulled back via this projection. Then on $\mathbb{C}_\lambda$, $\mathfrak{n}^+$ acts by 0. We can also regard $\mathbb{C}_\lambda$ as a $U(\mathfrak{b})$-module.

Definition 4.12. *Let $\lambda \in \mathfrak{h}^*$ be given. Define*

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

a left $U(\mathfrak{g})$-module under the natural action (left multiplication) of $U(\mathfrak{g})$. This is called the Verma module of highest weight λ.

Notice that $M(\lambda) \cong U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ as left $U(\mathfrak{n}^-)$-modules. This follows again from the triangular decomposition of $U(\mathfrak{g})$. In other words, $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank 1.

Define the vector $v = 1 \otimes 1 \in M(\lambda)$. This is a highest weight vector of weight λ and $M(\lambda) = U(\mathfrak{n}^-) \cdot v$. This means that $M(\lambda)$ is indeed a highest weight module of weight λ.

Lemma 4.13. *The weights of $M(\lambda)$ are $\Psi(M(\lambda)) = \lambda - Q^+$.*

Proof. We know that $\Psi(M(\lambda)) \subseteq \lambda - Q^+$ since we've seen that this holds for any highest weight module. The equality follows from the fact that $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank 1, hence all elements $e_{-\alpha_1}^{i_1} \cdots e_{-\alpha_m}^{i_m} \cdot v$ are linearly independent and they "cover" all the weights in $\lambda - Q^+$. □

An alternative definition goes as follows. Define the left ideal I of $U(\mathfrak{g})$ generated by $\mathfrak{n}^+$ and $\{h - \lambda(h) \cdot 1 \mid h \in \mathfrak{h}\}$. Then

$$M(\lambda) \cong U(\mathfrak{g})/I. \tag{4.4.1}$$

Lemma 4.14 (Universal property)**.** *Suppose M is a highest weight module of weight λ. Then there exists a projection $p : M(\lambda) \to M$.*

Proof. Let $v' \in M$ be a highest weight vector. The assignment $v \mapsto v'$ extends to a $U(\mathfrak{g})$-homomorphism $M(\lambda) \to M$ which is surjective since v' generates M.

Or differently, start with the projection $\widetilde{p} : U(\mathfrak{g}) \to M$, $1 \mapsto v$. Since the ideal I kills M, $\widetilde{p}$ factors through $U(\mathfrak{g})/I \to M$. □

By the properties of highest weight modules, $M(\lambda)$ has a unique maximal submodule $N(\lambda)$ and a unique simple quotient $L(\lambda) = M(\lambda)/N(\lambda)$.

Theorem 4.15. *Every simple module in $\mathcal{O}$ is isomorphic to a module $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. Moreover,*

$$\dim \operatorname{Hom}_{U(\mathfrak{g})}(L(\lambda), L(\mu)) = \begin{cases} 1, & \lambda = \mu \\ 0, & \lambda \neq \mu. \end{cases}$$

Proof. Every simple module L in $\mathcal{O}$ is a highest weight module of weight λ where λ is a maximal weight in $\Psi(L)$. By the universal property, L is a quotient of the Verma module of $M(\lambda)$, hence $L \cong L(\lambda)$. The second claim is just a particular case of Proposition 4.10. □

Remark 4.16. *Every simple module in category $\mathcal{O}$ is uniquely determined by its highest weight. In particular, this is true for simple finite dimensional $\mathfrak{g}$-modules.*

4.5. **Finite dimensional modules.** In light of the results in the previous subsection, we have a one-to-one correspondence

$$\mathfrak{h}^* \longrightarrow \{\text{simple modules in } \mathcal{O}\}, \quad \lambda \mapsto L(\lambda).$$

We would like to determine which modules $L(\lambda)$ are finite dimensional.

Recall that for every $\alpha \in \Phi^+$, we have the Lie triple $\{e_\alpha, h_\alpha, e_{-\alpha}\}$. Denote by $\mathfrak{sl}_\alpha$ the span of this triple. This is a Lie algebra isomorphic to $\mathfrak{sl}(2)$, and we will make use repeatedly of our knowledge of $\mathfrak{sl}(2)$-representation theory applied to $\mathfrak{sl}_\alpha$.

Theorem 4.17. (1) *The simple module $L(\lambda) \in \mathcal{O}$ is finite dimensional if and only if $\lambda \in P^+ = \{\chi \in \mathfrak{h}^* \mid \chi(h_\alpha) \geq 0,$ for all $\alpha \in \Phi^+\}$.*

(2) *This is the case if and only if $\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)}$, for all $\mu \in \Psi(L(\lambda)$ and all $w \in W$.*

Proof of the necessary condition in (1). It is easy to see that if $L(\lambda)$ is finite dimensional, then $\lambda \in P^+$. Indeed, let $\alpha \in \Phi^+$ be arbitrary and regard $L(\lambda)$ as a finite dimensional $\mathfrak{sl}_\alpha$-module. If v is a highest weight vector of $L(\lambda)$ with highest weight λ, then in particular, $e_\alpha \cdot v = 0$ and $h_\alpha \cdot v = \lambda(h_\alpha)v$. This means that v is a highest weight vector for a finite dimensional $\mathfrak{sl}_\alpha$-module, and therefore $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$. □

For the rest of the proof, we need to analyze first the structure of $M(\lambda)$. More precisely, if μ is a weight of $M(\lambda)$ then $\mu \leq \lambda$, i.e., $\mu \in \lambda - Q^+$. We look for highest weight vectors in $M(\lambda)$ with weight $\mu < \lambda$. (We have seen this idea already in the case of $\mathfrak{sl}(2)$.) The key calculation is in the following proposition.

Proposition 4.18. *Let $M(\lambda)$ be a Verma module, $\lambda \in \mathfrak{h}^*$, and let $v \in M(\lambda)$ be a highest weight vector of weight λ. Let α be a simple root. If $n := \lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$, then $e_{-\alpha}^{n+1} \cdot v$ is a highest weight vector of weight $\mu = \lambda - (n+1)\alpha < \lambda$.*

Proof. Denote $v' = e_{-\alpha}^{n+1} \cdot v$. Then

$$\begin{aligned} h \cdot v' &= h e_{-\alpha}^{n+1} \cdot v = [h, e_{-\alpha}^{n+1}] \cdot v + e_{-\alpha}^{n+1} h \cdot v \\ &= -\alpha(h)(n+1) e_{-\alpha}^{n+1} \cdot v + \lambda(h) e_{-\alpha}^{n+1} h \cdot v = (\lambda - (n+1)\alpha)(h) v'. \end{aligned}$$

This shows that v' is a μ-weight vector. Next we need to check that $\mathfrak{n}^+ \cdot v' = 0$. It is sufficient to verify that $e_\beta \cdot v' = 0$ for all $\beta \in \Pi$. (Every positive root vector can be written as a repeated commutator of simple root vectors.)

If $\beta \neq \alpha$, then $e_\beta \cdot v' = e_\beta \cdot e_{-\alpha}^{n+1} \cdot v = e_{-\alpha}^{n+1} e_\beta \cdot v = 0$. Here we used that e_α and e_β commute, because $\beta - \alpha$ is not a root.

If $\beta = \alpha$, then $e_\alpha \cdot v' = e_\alpha e_{-\alpha}^{n+1} \cdot v = [e_\alpha, e_{-\alpha}^{n+1}] \cdot v + e_{-\alpha}^{n+1} e_\alpha \cdot v = (n+1) e_{-\alpha}^{n} (h_\alpha - n) \cdot v + 0 = 0$, since $\lambda(h_\alpha) = n$. □

Corollary 4.19. *In the notation of Proposition 4.18, if $n = \lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$ then there exists an injective homomorphism $M(\mu) \to M(\lambda)$, $\mu = \lambda - (n+1)\alpha$, whose image lies in the unique maximal submodule of $M(\lambda)$.*

Proof. If v' is a highest weight vector of $M(\mu)$ of weight μ, then the homomorphism is defined by sending $v' \mapsto e_{-\alpha}^{n+1} \cdot v$, where v is a highest weight vector of weight λ in $M(\lambda)$ and extending as a $U(\mathfrak{g})$-homomorphism. The fact that this is injective follows from the fact that the two Verma modules are free of rank one over $U(\mathfrak{n}^-)$.

Since the image of the homomorphism is a submodule of $M(\lambda)$, it must lie in the unique maximal submodule. □

Remark 4.20. *The condition on μ in Proposition 4.18 becomes more transparent if we notice that $\mu = s_\alpha(\lambda + \rho) - \rho$. Thus the condition can be rephrased as $\mu := s_\alpha(\lambda + \rho) < \lambda + \rho$. The shift by ρ will make sense when we discuss the Harish-Chandra homomorphism later.*

Before we return to the proof of Theorem 4.17, we need one more definition and a lemma.

Definition 4.21. *Let $\mathfrak{a}$ be an algebra. An $\mathfrak{a}$-module M is called $\mathfrak{a}$-finite if it is a sum of finite dimensional $\mathfrak{a}$-modules.*

We have already encountered this notion in axiom ($\mathcal{O}3$) for category $\mathcal{O}$. The idea behind $\mathfrak{a}$-finite module is that we can extend to this setting the local properties of finite dimensional modules.

Lemma 4.22. *Let $\alpha \in \Pi$ be given and suppose that $M \in \mathcal{O}$ is an $\mathfrak{sl}_\alpha$-finite module. Then $\dim M_\mu = \dim M_{s_\alpha(\mu)}$ for every weight μ of M.*

Proof. Decompose $M = \bigoplus_{k \in \mathbb{Z}} M_k$ with respect to the action of h_α. Here M_k is the k-eigenspace of h_α. From the representation theory of finite dimensional $\mathfrak{sl}(2)$-modules (applied to $\mathfrak{sl}_\alpha$) we know that e_α^k induces a linear isomorphism between the k-eigenspace and the $(-k)$-eigenspace of a finite dimensional module. But then this is also true for $\mathfrak{sl}_\alpha$-modules so it can be applied to M. Denote $j_k : M_k \to M_{-k}$ the resulting linear isomorphism induced by the action of $e_{-\alpha}^k$. We can decompose

$$M_k = \bigoplus_{\mu \in \mathfrak{h}^*, \mu(h_\alpha) = k} M_\mu, \quad M_{-k} = \bigoplus_{\mu' \in \mathfrak{h}^*, \mu'(h_\alpha) = -k} M_{\mu'} = \bigoplus_{\mu \in \mathfrak{h}^*, \mu(h_\alpha) = k} M_{s_\alpha(\mu)}.$$

Now, if $v_\mu \in M_\mu$, we have $h \cdot e_{-\alpha}^k \cdot v_\mu = [h, e_{-\alpha}^k] \cdot v_\mu + e_{-\alpha}^k h \cdot v_\mu = (\mu - k\alpha)(h) e_{-\alpha}^k \cdot v_\mu = s_\alpha(\mu) e_{-\alpha}^k \cdot v_\mu$, since $k = \mu(h_\alpha)$. This shows that j_k maps M_μ to $M_{s_\alpha(\mu)}$. But then it has to induce a linear isomorphism between M_μ to $M_{s_\alpha(\mu)}$. □

Now we are in position to finish the proof of Theorem 4.17.

Proof of Theorem 4.17. Suppose $\lambda \in P^+$ and fix an arbitrary $\alpha \in \Pi$. Then $n = \lambda(h_\alpha)$ is a nonnegative integer. Regard $L(\lambda)$ as an $\mathfrak{sl}_\alpha$-module. We claim that $L(\lambda)$ is $\mathfrak{sl}_\alpha$-finite. To see this, denote by L' the sum of all the finite dimensional $\mathfrak{sl}_\alpha$-submodules of $L(\lambda)$.

If v is a highest weight vector of $L(\lambda)$, v is a highest weight vector for the $\mathfrak{sl}_\alpha$-action and it has weight $n \in \mathbb{Z}_{\geq 0}$. This implies that $\mathfrak{sl}_\alpha \cdot v$ is a finite dimensional $\mathfrak{sl}_\alpha$-module (of dimension $n+1$). Thus $\mathfrak{sl}_\alpha \cdot v \subset L'$ and $L' \neq 0$.

Next, we see that L' is a $\mathfrak{g}$-submodule of $L(\lambda)$. This is a completely general argument. Take $m' \in L'$ and $x \in \mathfrak{g}$, we want to see that $x \cdot m' \in L'$. By definition, there exists a finite dimensional $\mathfrak{sl}_\alpha$-submodule N of $L(\lambda)$ such that $m' \in N$. Since $\mathfrak{g}$ itself is a finite dimensional $\mathfrak{sl}_\alpha$-module (under the adjoint action), it follows that $\mathfrak{g} \cdot N$ is a finite dimensional $\mathfrak{sl}_\alpha$ module, and it is a submodule of $L(\lambda)$. Since $x \cdot m' \in \mathfrak{g} \cdot N$, it follows that $x \cdot m' \in L'$. Since L' is a nonzero $\mathfrak{g}$-submodule of $L(\lambda)$, it follows that $L(\lambda) = L'$ since $L(\lambda)$ is simple.

We can apply then Lemma 4.22 to $L(\lambda)$ to find that $\dim L(\lambda)_\mu = \dim L(\lambda)_{s_\alpha(\mu)}$ for all $\mu \in \Psi(L(\lambda))$. Since α was arbitrary and the s_αs generate W, it follows that $\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)}$ for all $\mu \in \Psi(L(\lambda))$.

Using this condition, we claim that there are only finitely many weights in $L(\lambda)$. Then the conclusion follows too since every weight space is finite dimensional. Recall that in every W-orbit on weights there exists one and only one dominant weight. Since W is finite, we only need to count the weights $\mu \in P^+$ that can appear in $L(\lambda)$. The second condition means that $\mu \leq \lambda$. But it is easy to see that there are only finitely many weights in P^+ below λ. (P is a lattice!)

□

Corollary 4.23. *Every finite dimensional $\mathfrak{g}$-module is a direct sum of simple modules $L(\lambda)$, $\lambda \in P^+$.*

Proof. This follows from the previous theorem and Weyl's theorem on complete reducibility.[5] □

We now have a classification via integral dominant highest weights of the simple finite dimensional $\mathfrak{g}$-modules. Other typical information that one would still like to have in representation theory is:

- the dimension of $L(\lambda)$,
- the formal character of $L(\lambda)$,
- models (or explicit realizations) of $L(\lambda)$.

We will obtain satisfactory answers for the first two topics, but the third topic, except for some particular examples, is beyond the scope of this course.

5. The center of $U(\mathfrak{g})$

Recall the notation $Z(\mathfrak{g})$ for the center of $U(\mathfrak{g})$. We wish to understand the structure of $Z(\mathfrak{g})$. We will first look at the action of $Z(\mathfrak{g})$ on modules in $\mathcal{O}$.

5.1. Infinitesimal characters.

Any algebra homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is called an *infinitesimal character*. Let $\Theta = \{\chi : Z(\mathfrak{g}) \to \mathbb{C}\}$ denote the set of all infinitesimal characters.

Lemma 5.1. *Let $M(\lambda) \in \mathcal{O}$ be the Verma module with $\lambda \in \mathfrak{h}^*$. Then every $z \in Z(\mathfrak{g})$ acts by a scalar on $M(\lambda)$.*

Proof. Again, we've already seen the following proof in the case of $\mathfrak{sl}(2)$. Write $M(\lambda) = U(\mathfrak{g}) \cdot v$, where v is a highest weight vector of weight λ. If $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$, then

$$h \cdot (z \cdot v) = zh \cdot v = \lambda(h)(z \cdot v),$$

meaning that $z \cdot v \in M(\lambda)_\lambda$ too. Since $\dim M(\lambda)_\lambda = 1$, there exists a scalar $c_z \in \mathbb{C}$ such that $z \cdot v = c_z v$. Now let $u \cdot v$, $u \in U(\mathfrak{g})$, be an arbitrary vector in $M(\lambda)$. Then

$$z \cdot (u \cdot v) = uz \cdot v = c_z u \cdot v.$$

This shows that z acts on $M(\lambda)$ by the scalar c_z. □

[5] Weyl's complete reducibility theorem is proved, using the action of the Casimir operator, in the Lie algebras course C2.1, see Kevin McGerty's notes.

The above lemma should remind us of Schur's Lemma from finite dimensional simple representations. Notice however that in our situation $M(\lambda)$ is not finite dimensional and in general it is not simple either. In light of this result, we define for each $\lambda \in fh^*$,

$$\chi_\lambda(z) = c_z, \tag{5.1.1}$$

in the notation of the proof of the lemma. This is an algebra homomorphism clearly, and we call it the infinitesimal character associated to λ.

We can say more about it. Suppose $z \in Z(\mathfrak{g})$ is given. Let $\lambda \in \mathfrak{h}^*$ and extend it to $\lambda : S(\mathfrak{h}) \to \mathbb{C}$ in the obvious way. Then using the triangular decomposition of $U(\mathfrak{g})$, we may write

$$z = \sum z_1 z^+ + z_0 + \sum z^- z_2,$$

where $z_1 \in U(\mathfrak{n}^-)U(\mathfrak{h})$, $z^+ \in U(\mathfrak{n}^+)$ has no constant terms, $z_0 \in U(\mathfrak{h}) = S(\mathfrak{h})$, $z^- \in U(\mathfrak{n}^-)$ has no constant terms, and $z_2 \in U(\mathfrak{h})$. Let $v \in M(\lambda)_\lambda$ be a highest weight vector. Then $z \cdot v = \lambda(z_0)v + \lambda(z_2)z^- \cdot v$. Since $z^- \cdot v$ consists of weight vectors of weight strictly smaller than λ, we see that $c_z = \lambda(z_0)$. Define

$$\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h}), \quad u \mapsto u^0, \tag{5.1.2}$$

the projection map defined by sending all monomials in positive degrees in $\mathfrak{n}^+$ and in $\mathfrak{n}^-$ to 0. This is a linear surjective map. The discussion before says that

$$\chi_\lambda(z) = \lambda(\mathrm{pr}(z)), \quad z \in Z(\mathfrak{g}). \tag{5.1.3}$$

Therefore, it makes sense to restrict pr to $Z(\mathfrak{g})$ and obtain the linear map

$$\xi' : Z(\mathfrak{g}) \to S(\mathfrak{h}). \tag{5.1.4}$$

This is called the *Harish-Chandra projection*.

Lemma 5.2. *The map $\xi' : Z(\mathfrak{g}) \to S(\mathfrak{h})$ is an algebra homomorphism.*

Proof. For $z_1, z_2 \in Z(\mathfrak{g})$, $\lambda(\mathrm{pr}(z_1 z_2)) = \chi_\lambda(z_1 z_2) = \chi_\lambda(z_1)\chi_\lambda(z_2) = \lambda(\mathrm{pr}(z_1))\lambda(\mathrm{pr}(z_2)) = \lambda(\mathrm{pr}(z_1)\,\mathrm{pr}(z_2))$. Hence $\mathrm{pr}(z_1 z_2) - \mathrm{pr}(z_1)\,\mathrm{pr}(z_2) \in \cap_{\lambda \in \mathfrak{h}^*} \ker \lambda = 0$. □

5.2. **The Harish-Chandra homomorphism.** To understand infinitesimal characters further, we need to know when $\chi_\lambda = \chi_\mu$ for $\lambda, \mu \in \mathfrak{h}^*$. From Lemma 5.1, we know that if $M(\mu)$ occurs as a submodule of $M(\lambda)$, then $\chi_\mu = \chi_\lambda$. We have already seen this situation: if $n = \lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$ for some $\alpha \in \Pi$, then $M(\mu)$ is a submodule of $M(\lambda)$ where $\mu = s_\alpha(\lambda + \rho) - \rho$. This fact motivates the following definition.

Definition 5.3. *Define the dot-action of W on $\mathfrak{h}^*$ by*

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \in W, \lambda \in \mathfrak{h}^*. \tag{5.2.1}$$

We say that λ and μ are linked if $\mu = w \cdot \lambda$ for some $w \in W$.

Notice that μ and λ are linked if and only if $\mu + \rho$ and $\lambda + \rho$ are in the same W-orbit in $\mathfrak{h}^*$ under the natural action of W. In particular, "linkage" is an equivalence relation on $\mathfrak{h}^*$.

Lemma 5.4. *If $\lambda \in P$ and μ is linked to λ, then $\chi_\lambda = \chi_\mu$.*

Proof. Since W is generated by simple reflections it is sufficient to consider the case when $\mu = s_\alpha \cdot \lambda$ for $\alpha \in \Pi$. If $\lambda \in P$, then $n = \lambda(h_\alpha) \in \mathbb{Z}$. If $n \geq 0$, by the result recalled before, $\chi_\mu = \chi_\lambda$. If $n = -1$, $\mu = \lambda$ and there is nothing to prove. If $n \leq -2$, then $\lambda = s_\alpha \cdot \mu$ and $\mu(h_\alpha) = -n - 2 \geq 0$, so we are back in the first case with the roles of λ and μ reversed. □

Proposition 5.5. *If $\lambda, \mu \in \mathfrak{h}^*$ are linked then $\chi_\lambda = \chi_\mu$.*

Proof. We know by the previous lemma that the proposition holds when λ and μ are in P.

Suppose $\mu = w \cdot \lambda$. Denote by $P(\mathfrak{h}^*)$ the algebra of polynomial functions on $\mathfrak{h}^*$ and by $\iota : S(\mathfrak{h}) \to P(\mathfrak{h}^*)$ the natural isomorphism. Then $\chi_\lambda(z) = \lambda(\xi'(z)) = \iota(\xi'(z))(\lambda)$. Similarly, $\chi_{w \cdot \lambda}(z) = (w \cdot \lambda)(\xi'(z)) = \lambda(w^{-1} \cdot \xi'(z)) = \iota(w^{-1} \cdot \xi'(z))(\lambda)$.

Therefore, $\iota(w^{-1} \cdot \xi'(z))(\lambda) = \iota(\xi'(z))(\lambda)$ for all $\lambda \in P$. Since $P \subset \mathfrak{h}^*$ is dense in the Zariski topology, it follows that $\iota(w^{-1} \cdot \xi'(z)) = \iota(\xi'(z))$ as polynomials on $\mathfrak{h}^*$, and so $\chi_\lambda = \chi_\mu$ for all $\lambda, \mu \in \mathfrak{h}^*$ that are linked. □

Definition 5.6. *The Harish-Chandra homomorphism is $\xi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ defined by the composition $\xi = \rho\text{-shift} \circ \xi'$ where $\xi' : Z(\mathfrak{g}) \to S(\mathfrak{h}) = P(\mathfrak{h}^*)$ is the Harish-Chandra projection and $\rho\text{-shift} : P(\mathfrak{h}^*) \to P(\mathfrak{h}^*)$ is the map defined by $p(\lambda) \mapsto p(\lambda - \rho)$ for all $p \in P(\mathfrak{h}^*)$, $\lambda \in \mathfrak{h}^*$.*

Proposition 5.7. *The image of the Harish-Chandra homomorphism $\xi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ lies in the subalgebra $S(\mathfrak{h})^W = P(\mathfrak{h}^*)^W$ of W-invariant polynomials.*

Proof. The definition of ξ means that $\chi_\lambda(z) = \xi(z)(\lambda + \rho)$, for all $\lambda \in \mathfrak{h}^*$. Here we think of $\xi(z) \in P(\mathfrak{h}^*)$. If λ and μ are linked, equivalently $\lambda + \rho$ and $\mu + \rho$ are W-conjugate with respect to the natural action of W, then $\chi_\mu = \chi_\lambda$. This means that $\xi(z)$ is constant on W-orbits in $\mathfrak{h}^*$, which is equivalent with the claim of the proposition. □

Remark 5.8. *A priori, the definition of ξ depends on the choice of positive roots. Using that any two choices of positive roots are conjugate under W, one may show that in fact ξ is independent of this choice.*

Theorem 5.9 (Harish-Chandra)**.** (1) *The algebra homomorphism $\xi : Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ is an isomorphism.*
(2) *If $\lambda, \mu \in \mathfrak{h}^*$, $\chi_\mu = \chi_\lambda$ if and only if λ and μ are linked.*
(3) *If $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is an infinitesimal character, then $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$.*

Remarks about the proof. (1) This is the hardest part. The essential step is *Chevalley's restriction theorem* which says that the restriction $\theta : P(\mathfrak{g}) \to P(\mathfrak{h})$, $\theta(f) = f\mid_{\mathfrak{h}}$ maps $P(\mathfrak{g})^{\mathfrak{g}}$ isomorphically onto $P(\mathfrak{h})^W$.

Since the algebra homomorphism ξ is compatible with the natural filtrations on $Z(\mathfrak{g})$ (inherited from $U(\mathfrak{g})$) and on $S(\mathfrak{h})^W$ (inherited from $S(\mathfrak{h})$), it is sufficient to check that the associated graded homomorphism $\bar{\xi}$: $\mathrm{gr}(Z(\mathfrak{g})) \to \mathrm{gr}S(\mathfrak{h})^W = S(\mathfrak{h})^W$ is an isomorphism. Recall that the symmetrizing map $\mathrm{symm} : S(\mathfrak{g}) \to \mathrm{gr}U(\mathfrak{g})$ induces an isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \cong \mathrm{gr}(Z(\mathfrak{g})) = \mathrm{gr}(U(\mathfrak{g})^{\mathfrak{g}})$. Writing $S(\mathfrak{g}) = S(\mathfrak{n}^-) \otimes S(\mathfrak{h}) \otimes S(\mathfrak{n}^+)$, denote by $p_0 : S(\mathfrak{g}) \to S(\mathfrak{h})$ the projection. One can check that

$$\bar{\xi} = p_0\mid_{S(\mathfrak{g})^{\mathfrak{g}}} .$$

Finally, one can apply the Chevalley restriction theorem using the identification $\mathfrak{h} \cong \mathfrak{g}^*$ and $\mathfrak{h} \cong \mathfrak{h}^*$ via the Killing form. ($S(\mathfrak{g}) = P(\mathfrak{g}^*) \cong P(\mathfrak{g})$ etc.)

(2) Suppose that λ and μ are in different linkage classes. Then $\lambda + \rho$ and $\mu + \rho$ are not W-conjugate. We will separate them using W-invariant polynomials on $\mathfrak{h}^*$. Since $W(\lambda + \rho)$ and $W(\mu + \rho)$ are finite sets, there exists a polynomial $f \in P(\mathfrak{h}^*)$ such that f takes the value 1 at every element of $W(\lambda + \rho)$ and the value 0 at every element of $W(\mu + \rho)$. To get a W-invariant polynomial, we average f:

$$\bar{f} = \sum_{w \in W} w(f).$$

Then $\bar{f} \in P(\mathfrak{h}^*)^W$ and $\bar{f}(\lambda + \rho) \neq 0 = \bar{f}(\mu + \rho)$. Set $z = \xi^{-1}(\bar{f}) \in Z(\mathfrak{g})$ and we have

$$\chi_\lambda(z) = \bar{f}(\lambda + \rho) \neq \bar{f}(\mu + \rho) = \chi_\mu(z).$$

(3) Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be an arbitrary infinitesimal character. Set $\widetilde{\chi} = \chi \circ \xi^{-1} : S(\mathfrak{h})^W \to \mathbb{C}$, an algebra homomorphism. One can show that $S(\mathfrak{h})$ is a finitely generated $S(\mathfrak{h})^W$-module. Then using standard facts in commutative algebra, it follows that every homomorphism $S(\mathfrak{h})^W|to\mathbb{C}$ can be extended to a homomorphism $S(\mathfrak{h}) \to \mathbb{C}$. But every homomorphism $S(\mathfrak{h}) = P(\mathfrak{h}^*) \to \mathbb{C}$ is given by evaluation at some λ. It follows that $\widetilde{\chi} = \chi_\lambda\mid_{S(\mathfrak{h})^W}$ and then $\chi = \chi_\lambda$. □

To conclude, recall that Θ denoted the set of all infinitesimal characters. The homomorphism $\xi : Z(\mathfrak{g}) \to P(\mathfrak{h}^*)^W$ gives rise to a dual map

$$\xi^* : \mathfrak{h}^*/W \to \Theta, \quad \xi^*(\lambda) = \chi_{\lambda - \rho}. \tag{5.2.2}$$

The Harish-Chandra Theorem says that ξ^* is a bijection.

Example 5.10. *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Recall the Casimir element $C = h^2 + 2h + 4fe \in Z(\mathfrak{g})$. The Harish-Chandra projection takes C to $\xi'(C) = h^2 + 2h$. Next, the ρ-shift takes $h \mapsto h - \rho(h) = h - 1$. This is because $\rho = \frac{1}{2}\alpha$ and $\alpha(h) = 2$. This means that the Harish-Chandra isomorphism ξ takes*

$$C \mapsto h^2 - 1.$$

In this case, $W = \{\mathrm{Id}, -\mathrm{Id}\} \cong S_2$, and therefore, the image of ξ is $S(\mathfrak{h})^W = \mathbb{C}[h^2]$. This means that $Z(\mathfrak{g}) = \mathbb{C}[C]$ and the correspondence is $1 \mapsto 1$ and $C \mapsto h^2 - 1$.

5.3. **Composition series.** Recall that we know that every $M \in \mathcal{O}$ is a Noetherian module.

Proposition 5.11. *Each $M \in \mathcal{O}$ is Artinian.*

Proof. We need to show that any proper descending chain of submodules of M terminates. In light of Corollary 4.11, it is sufficient to prove the claim when $M = M(\lambda)$. Suppose that $N' \subsetneq N$ are submodules of M. Since $Z(\mathfrak{g})$ acts by the infinitesimal character χ_λ on $M(\lambda)$, it acts by the same χ_λ on N/N'. The subquotient N/N' has a highest weight vector of weight $\mu \leq \lambda$, which implies that $\chi_\mu = \chi_\lambda$, and so μ is in the same linkage class of λ. Moreover, μ must be a weight of N and $\dim N_\mu > \dim N'_\mu$.

Define $V = \sum_{w\in W} M(\lambda)_{w\cdot\lambda}$. Since all weight spaces are finite dimensional, $\dim V < \infty$. Then $N \cap V \neq 0$ and

$$\dim(N \cap V) > \dim(N' \cap V) > 0.$$

Therefore, any properly descending chain of submodules must terminate. □

Corollary 5.12. *Every $M \in \mathcal{O}$ has a finite composition series with simple factors isomorphic to $L(\lambda)$s, $\lambda \in \mathfrak{h}^*$.*

Proof. Since $M \in \mathcal{O}$ is both Artinian and Noetherian, the Jordan-Hölder Theorem applies, hence M has a finite composition series. The second claim is immediate since all the simple modules in $\mathcal{O}$ are of the form $L(\lambda)$. □

If $L(\mu)$ is a factor in a composition series of M, denote by $[M : L(\mu)]$ the multiplicity with which $L(\mu)$ appears. Recall that this does not depend, nor do the $L(\mu)$s, on the composition series of M!

Corollary 5.13. *If $M(\lambda)$ is a Verma module, and $L(\mu)$ is a factor of $M(\lambda)$, then $\mu \leq \lambda$ and λ, μ are linked. In this case, $[M(\lambda : L(\mu)] < \infty$ and $[M(\lambda) : L(\lambda)] = 1$.*

5.4. **The Grothendieck group.** The definition of the Grothendieck group applies to every small abelian category. Define A to be the free abelian group generated by symbols $[M]$, where M ranges over isomorphism classes of $M \in \mathcal{O}$. Define B to be the subgroup of A generated by expressions $[M_1] + [M_2] - [M]$ for all exact sequences

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0.$$

Definition 5.14. *The Grothendieck group is $K(\mathcal{O}) = A/B$.*

Since every object in $\mathcal{O}$ has finite length, one can show that the set of (isomorphism classes of) irreducible modules $\mathsf{Irr}\ \mathcal{O} = \{[L(\lambda)] \mid \lambda \in \mathfrak{h}^*\}$ is a $\mathbb{Z}$-basis of $K(\mathcal{O})$.

Proposition 5.15. *The set of "standard modules" $\mathsf{St}\ \mathcal{O} = \{[M(\lambda) \mid \lambda \in \mathfrak{h}^*\}$ is also a $\mathbb{Z}$-basis of $K(\mathcal{O})$. The change of basis matrix between $\mathsf{St}\ \mathcal{O}$ and $\mathsf{Irr}\ \mathcal{O}$ is upper triangular with 1s on the diagonal.*

Proof. The second claim implies the first since the inverse of an upper triangular matrix with integer entries and 1s on the diagonal is of the same form. If $L(\mu)$ is any simple factor of $M(\lambda)$, then $\chi_\mu = \chi_\lambda$, hence λ and μ are linked. Moreover, $\mu \leq \lambda$ as it is a weight of $M(\lambda)$. This means that

$$[M(\lambda)] = [L(\lambda)] + \sum_{\mu<\lambda,\ \mu\in W\cdot\lambda} n_{\lambda,\mu}[L(\mu)], \tag{5.4.1}$$

for certain nonnegative integers $n_{\lambda,\mu}$.

A better way to think about this is that first one partitions the sets $\{[M(\lambda)]\}$ and $\{[L(\lambda)]\}$ according to the orbits of the dot action of W on $\mathfrak{h}^*$. Then for each such orbit, the number of $[M(\lambda)]$ (respectively $[L(\lambda)]$) is finite and the change of basis matrix is finite, square, integer-valued, and upper triangular with 1 on the diagonal. Hence the inverse of the matrix has the same properties. □

Remark 5.16. *The setting where the Grothendieck group has two bases, one consisting of irreducibles and the other of standard modules, which are related via an upper uni-triangular matrix appears frequently in representation theory. You may have already seen this in the case of complex finite dimensional representations of the symmetric group S_n. In that case, both bases are indexed by partitions of n, with the standard modules being the induced modules from the trivial representation of the parabolic subgroups given by the rows of the partitions.*

6. Character formulas

The goal is to obtain the Weyl character formula and the dimension formula for a finite dimensional simple module $L(\lambda)$, $\lambda \in P^+$. We follow the exposition in Bernstein's notes.

6.1. **Formal characters.** Recall that the formal character of a module $M \in \mathcal{O}$ is the function

$$\mathsf{ch}_M : \mathfrak{h}^* \to \mathbb{Z}, \quad \mathsf{ch}_M(\lambda) = \dim M_\lambda < \infty.$$

Define the Kostant partition function $K : \mathfrak{h}^* \to \mathbb{Z}$ by $K(\mu) =$ the number of ways in which μ can be written as $\mu = \sum_{\alpha\in\Phi^+} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}_{\geq 0}$. Then, as seen before,

$$\dim M(\lambda)_\mu = K(\lambda - \mu), \quad \mu \in \mathfrak{h}^*. \tag{6.1.1}$$

It is convenient for the later calculations to consider instead the negative Kostant partition function:

$$p : \mathfrak{h}^* \to \mathbb{Z}, \quad p(\mu) = K(-\mu). \tag{6.1.2}$$

Notice that

$$p = \mathsf{ch}_{M(0)}.$$

For every function $f : \mathfrak{h}^* \to \mathbb{Z}$, define the support of f to be

$$\mathsf{supp} f = \{\chi \in \mathfrak{h}^* \mid f(\chi) \neq 0\}.$$

Define $\mathcal{E}$ to be the set of functions $f : \mathfrak{h}^* \to \mathbb{Z}$ such that $\mathsf{supp} f$ is contained in a finite union of sets of the form $\lambda_i - Q^+$.

Example 6.1. *For every $\mu \in \mathfrak{h}^*$, define the delta function at μ to be $\delta_\mu \in \mathcal{E}$, $\delta_\mu(\chi) = 1$ if $\mu = \chi$ and 0, if $\mu \neq \chi$.*

Recall that $\mathsf{supp}\ \mathsf{ch}_{M(\lambda)} = \lambda - Q^+$ and, in particular, $\mathsf{supp}\ p = -Q^+$. The set $\mathcal{E}$ can be endowed with the convolution product, for $f, g \in \mathcal{E}$:

$$(f \star g)(\mu) = \sum_{\chi\in\mathfrak{h}^*} f(\chi) g(\mu - \chi) = \sum_{\chi\in\mathfrak{h}^*} f(\mu - \chi) g(\chi). \tag{6.1.3}$$

Because of the support condition on the elements of $\mathcal{E}$, there are only finitely many nonzero elements in the sum, hance the convolution is well defined.

Lemma 6.2. (1) *$(\mathcal{E}, +, \star)$ is an associative and commutative ring with identity δ_0.*
(2) *$\delta_\mu \star \delta_\lambda = \delta_{\mu+\lambda}$, for all $\mu, \lambda \in \mathfrak{h}^*$.*

Proof. Straightforward. □

Lemma 6.3. $\mathsf{ch}_{M(\lambda)} = p \star \delta_\lambda$.

Proof. By the result recalled before, $\dim M(\lambda)_\mu = K(\lambda - \mu) = p(\mu - \lambda)$. On the other hand, $(p \star \delta_\lambda)(\mu) = \sum_{\chi\in\mathfrak{h}^*} p(\mu - \chi)\delta_\lambda(\chi) = p(\mu - \lambda)$. □

In light of this formula, it is desirable to find the inverse (if it exists) of p in $\mathcal{E}$.

Definition 6.4 (Weyl's denominator)**.** *Set $\Delta = \prod_{\alpha\in\Phi^+}(\delta_{\alpha/2} - \delta_{-\alpha/2}) \in \mathcal{E}$. Here $\prod$ means the convolution product and the order is not important because of the commutativity of $\mathcal{E}$. Notice that $\Delta = \prod_{\alpha\in\Phi^+} \delta_{\alpha/2}(\delta_0 - \delta_{-\alpha}) = \delta_\rho \prod_{\alpha\in\Phi^+}(\delta_0 - \delta_{-\alpha})$*

Lemma 6.5. *We have $p \star \Delta \star \delta_{-\rho} = \delta_0$, or equivalently $p \star \Delta = \delta_\rho$.*

Proof. Set $a_\alpha = \delta_0 + \delta_{-\alpha} + \delta_{-2\alpha} + \dots + \delta_{-n\alpha} + \dots$ for every $\alpha \in \Phi^+$. Then $p = \prod_{\alpha>0} a_\alpha$. Next notice that $a_\alpha \star (\delta_0 - \delta_{-\alpha}) = a_\alpha - a_\alpha \star \delta_{-\alpha} = \delta_0$. The claim follows from the formula of Δ. □

Theorem 6.6. *Define the map $\tau : K(\mathcal{O}) \to \mathcal{E}$ by $\tau([M]) = \mathsf{ch}_M \star \Delta$ for all $M \in \mathcal{O}$. Then:*
(1) *$\tau([M(\lambda)] = \delta_{\lambda+\rho}$.*
(2) *τ gives an isomorphism of $K(\mathcal{O})$ onto the subgroup $\mathcal{E}_c$ of $\mathcal{E}$ consisting of functions with finite support.*

Proof. For (1), $\tau([M(\lambda)] = \mathsf{ch}_{M(\lambda)} \star \Delta = \delta_\lambda \star p \star \Delta = \delta_\lambda \star \delta_\rho = \delta_{\lambda+\rho}$.

For (2), recall that $\{[M(\lambda)] \mid \lambda \in \mathfrak{h}^*\}$ is a $\mathbb{Z}$-basis of $K(\mathcal{O})$. By (1), τ induces an isomorphism onto the span of $\{\delta_\mu \mid \mu \in \mathfrak{h}^*\}$, but this is the same as $\mathcal{E}_c$. □

6.2. **Characters of simple finite dimensional modules.** The Weyl group W acts on $\mathfrak{h}^*$ via the natural action $w(\chi)$, $w \in W$, $\chi \in \mathfrak{h}^*$. This induces an action, the left regular action, on functions, i.e., on $\mathcal{E}$ via

$$(wf)(\chi) = f(w^{-1}(\chi)),\ w \in W,\ f \in \mathcal{E}.$$

Let $\det(w)$ denote the determinant of $w \in W$ viewed as a linear transformation of $\mathfrak{h}^*$.

Lemma 6.7. *The function Δ is W-skew-invariant, i.e., $w\Delta = \det(w) \cdot \Delta$, for all $w \in W$.*

Proof. It is sufficient to check that for every simple root α, $s_\alpha \Delta = -\Delta$. (Recall that $\det(s_\alpha) = -1$.) We know that s_α permutes the roots $\Phi^+ \setminus \{\alpha\}$ and $s_\alpha(\alpha) = -\alpha$. Hence

$$s_\alpha \Delta = \prod_{\beta \in \Phi^+} s_\alpha(\delta_{\beta/2} - \delta_{-\beta/2}) = (\delta_{-\alpha/2} - \delta_{\alpha/2}) \star \prod_{\beta \in \Phi^+ \setminus \{\alpha\}} (\delta_{\beta/2} - \delta_{-\beta/2}) = -\Delta.$$

□

Theorem 6.8. *Suppose that $L(\lambda)$ is finite dimensional, i.e., $\lambda \in P^+$. Then*

$$\Delta \star \mathsf{ch}_{L(\lambda)} = \sum_{w \in W} \det(w) \cdot \delta_{w(\lambda+\rho)}.$$

Proof. By inverting the formulas (5.4.1), we see that in $K(\mathcal{O})$, we have

$$[L(\lambda)] = \sum_{w \in W} m_{\lambda,w} [M(w \cdot \lambda)],$$

for some integers (not necessarily nonnegative) $m_{\lambda,w}$. Moreover, $m_{\lambda,1} = 1$. Since $\lambda \in P^+$, the condition $\mu = w \cdot \lambda < \lambda$ if $w \neq 1$ is automatic, which explains its absence in this formula.

Apply τ to this identity and get

$$\Delta \star \mathsf{ch}_{L(\lambda)} = \sum_{w \in W} m_{\lambda,w} \delta_{w(\lambda+\rho)},$$

since $w \cdot \lambda + \rho = w(\lambda + \rho)$. It remains to show that $m_{\lambda,w} = \det(w)$.

We claim that the left hand side of the identity is W-skew-invariant. Indeed, $w(\Delta \star \mathsf{ch}_{L(\lambda)}) = w\Delta \star w\mathsf{ch}_{L(\lambda)} = \det(w)\Delta \star w\mathsf{ch}_{L(\lambda)}$. But we proved that when $L(\lambda)$ is finite dimensional, $\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)}$ for all w, or in other words, $w\mathsf{ch}_{L(\lambda)} = \mathsf{ch}_{L(\lambda)}$. So, $w(\Delta \star \mathsf{ch}_{L(\lambda)}) = \det(w)\Delta \star \mathsf{ch}_{L(\lambda)}$. But then the right hand side of the identity is also W-skew-invariant, and since $m_{\lambda,1} = 1$, it follows that $m_{\lambda,w} = \det(w)$. □

Corollary 6.9 (Weyl's denominator formula)**.** $\prod_{\alpha \in \Phi^+} (\delta_{\alpha/2} - \delta_{-\alpha/2}) = \Delta = \sum_{w \in W} \det(w) \cdot \Delta_{w(\rho)}$.

Proof. This is the case $\lambda = 0$ in the previous theorem. □

Corollary 6.10. *Suppose $\lambda \in P^+$.*

(1) *(BGG formula) In the Grothendieck group $K(\mathcal{O})$, we have*

$$[L(\lambda)] = \sum_{w \in W} \det(w)\ [M(w \cdot \lambda)]. \tag{6.2.1}$$

(2) *(Kostant's multiplicity formula)*

$$\mathsf{ch}_{L(\lambda)}(\mu) = \sum_{w \in W} \det(w) K(w \cdot \lambda - \mu) = \sum_{w \in W} \det(w) K(w(\lambda + \rho) - (\mu + \rho)). \tag{6.2.2}$$

Proof. (1) follows from Theorem 6.8 since τ is an isomorphism. (2) is immediate from (1) given the formula for $\mathsf{ch}_{M(\lambda)}$ in terms of K.

□

Remark 6.11. *Formula (6.2.1) says that we can describe easily the columns corresponding to $\lambda \in P^+$ of the matrix that gives $\mathsf{Irr}\ \mathcal{O}$ in terms of $\mathsf{St}\ \mathcal{O}$. In general however, describing all the entries of this matrix is a difficult problem which is resolved by the Kazhdan-Lusztig conjectures, the most basic case being that of $\lambda = 0$.*

Example 6.12. *Let us consider the case of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. We identify, as we may, P^+ with $\mathbb{Z}_{\geq 0}$ and then $\alpha = 2\rho$ is identified with $2 \in \mathbb{Z}_{\geq 0}$. The Weyl group is $\{\pm 1\}$. Let $L(n)$, $n \in \mathbb{Z}_{\geq 0}$ be the simple module of dimension $n+1$. The weights of $L(n)$ are $n, n-2, n-4, \dots, -n$. This means that in $\mathcal{E}$ (identified with the space of functions $f : \mathbb{C} \to \mathbb{Z}$), the character of $L(n)$ can be expressed as:*

$$\mathsf{ch}_{L(n)} = \delta_n + \delta_{n-2} + \cdots + \delta_{-n+2} + \delta_{-n}.$$

On the other hand $\Delta = \delta_1 - \delta_{-1}$. Therefore, Theorem 6.8 becomes the easy identity

$$(\delta_n + \delta_{n-2} + \cdots + \delta_{-n+2} + \delta_{-n}) \star (\delta_1 - \delta_{-1}) = \delta_{n+1} - \delta_{-(n+1)}.$$

The formula (6.2.1) in $K(\mathcal{O})$ in this case is:

$$[L(n)] = [M(n)] - [M(-n-2)].$$

Notice that $w \cdot n = -n-2$ when $w = -1 \in W$. This formula is simply encoding in the Grothendieck group the fact that we have the short exact sequence

$$0 \longrightarrow M(-n-2) \longrightarrow M(n) \longrightarrow L(n) \longrightarrow 0.$$

6.3. **Weyl's character formula.** We wish to use the formula in Theorem 6.8 to deduce the dimension formula for a finite dimensional $L(\lambda)$, $\lambda \in P^+$. For this, we need to rephrase it in an analytic way, as in the original formula due to H. Weyl, which would allow us to apply a l'Hôpital rule argument.

For every $\chi \in \mathfrak{h}^*$, define $e^\chi = \sum_{i \geq 0} \frac{\chi^i}{i!}$. This is a formal power series in $\mathfrak{h}^*$. Denote the ring of formal power series by $\mathbb{C}[[\mathfrak{h}^*]]$.

Lemma 6.13. *The map $j : \mathcal{E}_c \to \mathbb{C}[[\mathfrak{h}^*]]$ given by $\delta_\lambda \mapsto e^\lambda$ is a ring homomorphism, where the multiplication in the left hand side is the convolution and on the right hand side is the multiplication of power series.*

Proof. Straightforward. □

The identification via j allows us to regard the character of every finite dimensional module M (in particular $M = L(\lambda)$, $\lambda \in P^+$) as an element of $\mathbb{C}[[\mathfrak{h}^*]]$:

$$\mathsf{ch}_M = \sum_{\mu \in \mathfrak{h}^*} (\dim M_\mu) e^\mu. \tag{6.3.1}$$

This is a finite sum since M is assumed finite dimensional. Let $\mathbb{C}(\mathfrak{h}^*)$ denote the field of fractions of $\mathbb{C}[[\mathfrak{h}^*]]$.

Theorem 6.14 (Weyl's character formula)**.** *Let $\lambda \in P^+$ be given. In $\mathbb{C}(\mathfrak{h}^*)$, the following identity holds:*

$$\mathsf{ch}_{L(\lambda)} = \frac{\sum_{w \in W} \det(w)\ e^{w(\lambda+\rho)}}{\sum_{w \in W} \det(w)\ e^{w(\rho)}}. \tag{6.3.2}$$

Proof. The identity in Theorem 6.8 is a formula in $\mathcal{E}_c$. Therefore, we may apply j to it and arrive to a corresponding identity in $\mathbb{C}[[\mathfrak{h}^*]]$. The desired formula then follows if we also take into account the Weyl denominator formula. □

Remark 6.15. *For every χ, the formal expression e^χ can be regarded as a function on $\mathfrak{h}$, $e^\chi : \mathfrak{h} \to \mathbb{C}$ via $e^\chi(h) = \sum_{i \geq 0} \frac{1}{i!} \chi(h)^i$. This is an analytic function $\mathfrak{h} \to \mathbb{C}$. This implies that both the numerator and the denominator in (6.3.2) are analytic functions in $\mathfrak{h}$. Moreover, $\mathsf{ch}_{L(\lambda)}$, $\lambda \in P^+$, being a finite sum of e^χs, it is also an analytic function on $\mathfrak{h}$. Therefore, one should regard formula (6.3.2) as an equality of analytic functions on $\mathfrak{h}$.*

Example 6.16. *In $\mathfrak{sl}(2,\mathbb{C})$, with the same notation and conventions as before, Weyl's character formula says that*

$$\mathsf{ch}_{L(n)}(x) = e^{nx} + e^{(n-2)x} + \cdots + e^{(-n+2)x} + e^{-nx} = \frac{e^{(n+1)x} - e^{-(n+1)x}}{e^x - e^{-x}}, \tag{6.3.3}$$

as functions $\mathfrak{h} \cong \mathbb{C} \to \mathbb{C}$.

6.4. **Weyl's dimension formula.** Regarding $\mathsf{ch}_{L(\lambda)}$, $\lambda \in P^+$, as an analytic function on $\mathfrak{h}$, it is clear that

$$\mathsf{ch}_{L(\lambda)}(0) = \sum_{\mu \in \mathfrak{h}^*} \dim L(\lambda)_\mu = \dim L(\lambda).$$

The idea then is to evaluate formula (6.3.2) at 0. The problem is that the fraction on the right hand side becomes an indeterminate of the form $\frac{0}{0}$, since $\sum_{w \in W} \det(w) = 0$, hence we need to use some form of l'Hôpital's rule.

Theorem 6.17. *The dimension of the finite dimensional simple module $L(\lambda)$, $\lambda \in P^+$ is*

$$\dim L(\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}. \tag{6.4.1}$$

Proof. To simplify notation, denote for each $\chi \in \mathfrak{h}^*$, $F_\chi : \mathfrak{h} \to \mathbb{C}$, $F_\chi = \sum_{w \in W} \det w \; e^{w(\chi)}$. Weyl's character formula becomes

$$\mathsf{ch}_{L(\lambda)} = \frac{F_{\lambda+\rho}}{F_\rho}$$

as analytic functions on $\mathfrak{h}$. Then

$$\mathsf{ch}_{L(\lambda)}(0) = \lim_{t \to 0} \frac{F_{\lambda+\rho}(th)}{F_\rho(th)},$$

where $h \in \mathfrak{h}$ has to be chosen so that $F_\rho(th) \neq 0$ for $t \neq 0$. The Weyl denominator formula in this setting says that

$$F_\rho = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}), \tag{6.4.2}$$

so we need to choose h such that $\alpha(th) = t\alpha(h) \neq 0$ for all $\alpha \in P^+$. Any h in the dominant Weyl chamber would do, but for reasons of convenience, we choose $h = \rho'$, where ρ' is the unique element of $\mathfrak{h}$ such that $\chi(\rho') = \langle \chi, \rho \rangle$ for all $\chi \in \mathfrak{h}^*$. Similarly, define λ' to be the unique element of $\mathfrak{h}$ such that $\chi(\lambda') = \langle \chi, \lambda \rangle$. In other words, recall that the Killing form restricted to $\mathfrak{h}$ induces an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$; then, λ' and ρ' are the preimages in $\mathfrak{h}$ of λ and ρ, respectively.

Thus, $\mathsf{ch}_{L(\lambda)}(0) = \lim_{t \to 0} \frac{F_{\lambda+\rho}(t\rho')}{F_\rho(t\rho')}$. Now

$$\begin{aligned} F_{\lambda+\rho}(t\rho') &= \sum_{w \in W} \det(w)\; e^{w(\lambda+\rho)}(t\rho') = \sum_{w \in W} \det(w)\; e^{(\lambda+\rho)(tw^{-1}(\rho'))} \\ &= \sum_{w \in W} \det(w)\; e^{t\langle \lambda+\rho, w^{-1}(\rho)\rangle} = \sum_{w \in W} \det(w)\; e^{\langle w^{-1}(\rho), t(\lambda+\rho)\rangle} \\ &= \sum_{w \in W} \det(w)\; e^{w^{-1}(\rho)}(t(\lambda' + \rho')) = F_\rho(t(\lambda' + \rho')), \end{aligned}$$

since $\det(w) = \det(w^{-1})$. By applying the Weyl denominator formula for F_ρ again, we find that

$$\dim L(\lambda) = \prod_{\alpha \in \Phi^+} \lim_{t \to 0} \frac{F_\rho(t(\lambda' + \rho'))}{F_\rho(t\rho')} = \prod_{\alpha \in \Phi^+} \lim_{t \to 0} \frac{e^{\frac{t}{2}\alpha(\lambda'+\rho')} - e^{-\frac{t}{2}\alpha(\lambda'+\rho')}}{e^{\frac{t}{2}\alpha(\rho')} - e^{-\frac{t}{2}\alpha(\rho')}}. \tag{6.4.3}$$

By L'Hôpital's rule, $\lim_{t \to 0} \frac{e^{\frac{t}{2}m} - e^{-\frac{t}{2}m}}{e^{\frac{t}{2}n} - e^{-\frac{t}{2}n}} = m/n$, which means that the factor for α contributes $\frac{\alpha(\lambda'+\rho')}{\alpha(\rho')} = \frac{\langle \alpha, \lambda+\rho \rangle}{\langle \alpha, \rho \rangle}$ to the product. The dimension formula is now proved. □

6.5. **The BGG resolution.** The identity (6.2.1) in $K(\mathcal{O})$ has a beautiful homological interpretation. The first step is to determine the maximal submodule $N(\lambda)$ of $M(\lambda)$ when $\lambda \in P^+$.

Proposition 6.18. *Suppose $\lambda \in P^+$. The maximal submodule $N(\lambda)$ of the Verma module $M(\lambda)$ is*

$$N(\lambda) = \sum_{\alpha \in \Pi} M(s_\alpha \cdot \lambda).$$

Proof. Since $\lambda \in P^+$, recall that we have $s_\alpha(\lambda+\rho) < \lambda+\rho$, or equivalently, $s_\alpha \cdot \lambda < \lambda$. By Corollary 4.19, $M(s_\alpha \cdot \lambda)$ is a submodule of $M(\lambda)$. This means that $\sum_{\alpha\in\Pi} M(s_\alpha\cdot\lambda) \subseteq N(\lambda)$. Define

$$A(\lambda) = M(\lambda)/\sum_{\alpha\in\Pi} M(s_\alpha\cdot\lambda). \tag{6.5.1}$$

This is sometimes called the *Weyl module* of weight λ. We want to show that

$$A(\lambda) \cong L(\lambda).$$

Notice that $A(\lambda)$ is a highest weight module (being a quotient of $M(\lambda)$) of highest weight λ. This is because λ is not a weight of any $M(s_\alpha\cdot\lambda)$. Let $v\neq 0$ be a highest weight vector in $M(\lambda)$ of weight λ and denote by $\bar v \neq 0$ the image of v in $A(\lambda)$. Then $A(\lambda)$ is generated by $\bar v$ as already remarked.

We claim that $\bar v$ is an $\mathfrak{sl}_\alpha$-finite vector for every $\alpha\in\Pi$. As in the proof of Proposition 4.18, the submodule $M(s_\alpha\cdot\lambda)$ contains (as a highest weight vector in fact) the vector $e_{-\alpha}^{\lambda(h_\alpha)+1}\cdot v$. This means that in $A(\lambda)$, $e_{-\alpha}^{\lambda(h_\alpha)+1}\cdot \bar v = 0$. Since $\bar v$ is preserved by h_α, killed by e_α and killed by a power of $e_{-\alpha}$, indeed, $\dim \mathfrak{sl}_\alpha\cdot\bar v<\infty$. Since $\bar v$ generates $A(\lambda)$, it follows that $A(\lambda)$ is an $\mathfrak{sl}_\alpha$-finite module. Applying Lemma 4.22, we obtain that $\dim A(\lambda)_\mu = \dim A(\lambda)_{s_\alpha(\mu)}$, for all μ. Since α was arbitrary, and s_α, $\alpha\in\Pi$ generate W, we see that the character of $A(\lambda)$ is W-invariant. The argument from Theorem 4.17 then implies that $A(\lambda)$ is finite dimensional.

By complete reducibility, $A(\lambda)=\bigoplus_{i=1}^k L(\lambda_i)$, where $\lambda_i\in P^+$, $1\le i\le k$. Since the center $Z(\mathfrak g)$ acts on $M(\lambda)$ by the infinitesimal character χ_λ, it also acts on $A(\lambda)$ and so on every $L(\lambda_i)$ by the same infinitesimal character χ_λ. Therefore, λ_i are all in the linkage class $W\cdot\lambda$. Since λ_i,λ are in P^+, this is only possible if $\lambda_i=\lambda$. So $A(\lambda)=L(\lambda_i)^{\oplus k}$. On the other hand, $\dim A(\lambda)_\lambda = 1$, so $k=1$. □

The proposition can be restated as follows.

Corollary 6.19. *If $\lambda\in P^+$, there exists an exact sequence*

$$\bigoplus_{\alpha\in\Pi} M(s_\alpha\cdot\lambda)\to M(\lambda)\to L(\lambda)\to 0.$$

The BGG resolution extends the sequence above to a resolution of $L(\lambda)$. We will state the theorem without proof. If interested, you may consult the proof in chapter 6 of Humphreys' book on category $\mathcal O$. To state it, we recall that W has a length function $\ell: W\to\mathbb Z_{\ge 0}$ such that $\ell(1)=0$, $\ell(s_\alpha)=1$ if $\alpha\in\Pi$, and $\ell(w)$ is the number of simple reflections in a (equivalently, any) minimal expression of w in terms of simple reflections. In particular,

$$\det(w) = (-1)^{\ell(w)}.$$

Alternatively, thinking in terms of Weyl chambers, $\ell(w)$ is the number of root hyperplanes that one needs to cross to go from the fundamental chamber $\mathcal C$ to the chamber $w\mathcal C$ via a shortest path.

Another equivalent definition is

$$\ell(w) = \#\{\alpha\in\Phi^+ \mid w(\alpha)\in\Phi^-\}. \tag{6.5.2}$$

The Weyl group W contains a unique element of maximal length, denoted w_0. It is a fact that $\ell(w_0)=|\Phi^+|$, in other words, $w_0(\alpha)\in\Phi^-$ for every $\alpha\in\Phi^+$.

Theorem 6.20 (BGG resolution)**.** *Suppose $\lambda\in P^+$. Then there exists an exact sequence:*

$$0\to M(w_0\cdot\lambda)\to \bigoplus_{\ell(w)=|\Phi^+|-1} M(w\cdot\lambda)\to\cdots\to\bigoplus_{\ell(w)=k} M(w\cdot\lambda)\to\cdots\to\bigoplus_{\alpha\in\Pi} M(s_\alpha\cdot\lambda)\to M(\lambda)\to L(\lambda)\to 0. \tag{6.5.3}$$

As a consequence, the Euler-Poincaré principle implies that in $K(\mathcal O)$:

$$[L(\lambda)] = \sum_{k=0}^{|\Phi^+|}(-1)^k\sum_{w\in W,\ell(w)=k}[M(w\cdot\lambda)] = \sum_{w\in W}(-1)^{\ell(w)}[M(w\cdot\lambda)], \tag{6.5.4}$$

which is exactly formula (6.2.1) established before using characters.

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