C2.3 Representations of semisimple Lie algebras

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Problem Sheet 1

1. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and recall the finite dimensional irreducible $\mathfrak{g}$-modules $V(n)$ of dimension $n+1$ constructed in the lectures. (Here, $n \geq 0$ is an integer.)

(i) Verify that $V(n) \cong S^n(V(1))$ as $\mathfrak{g}$-modules, where S^n denotes the n-th symmetric power.

(ii) Decompose the tensor product $V(2) \otimes V(3)$ into a direct sum of simple $\mathfrak{g}$-modules.

(iii) For $n \geq m \geq 0$, prove that

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m).$$

(iv) Find the decomposition of $S^n(V(2))$ into a direct sum of irreducible representations.

[*Hint: use the eigenspace decomposition with respect to the action of $h \in \mathfrak{g}$.*]

2. Let $\mathfrak{g} = \mathfrak{sl}(n)$ and V be the standard n-dimensional $\mathfrak{g}$-representation. For $1 \leq i \leq n-1$, denote $U_i = \bigwedge^i V$.

(a) Show that each U_i is a simple $\mathfrak{g}$-module and determine its highest weight.

(b) Deduce that any simple finite dimensional $\mathfrak{g}$-module appears as a summand of $S^{m_1}(U_1) \otimes S^{m_2}(U_2) \otimes \cdots \otimes S^{m_{n-1}}(U_{n-1})$, for some nonnegative integers m_i.

3. Suppose the characteristic of the field k is $p > 0$, and let $\mathfrak{g} = sl(2, \mathsf{k})$ and let $V(n)$ be the $\mathfrak{g}$-modules of dimension $n+1$ as before. Prove that $V(n)$ is irreducible as long as $n < p$, but reducible when $n = p$.

Let $\mathfrak{g}$ is a semisimple complex Lie algebra, $\mathfrak{h}$ is a Cartan subalgebra, Φ is the system of roots of $\mathfrak{h}$ in $\mathfrak{g}$, Φ^+ is a choice of positive roots, and Π is a corresponding base of simple roots. As in the lecture notes, denote by P the weight lattice and by Q the root lattice.

4 (*optional*)**.** Recall the weight $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ defined in the lectures. Show that:

(a) For every $\alpha \in \Pi$, $s_\alpha(\rho) = \rho - \alpha$.

(b) For every $\alpha \in \Pi$, $\rho(h_\alpha) = 1$, where h_α is the coroot of α.

(c) Suppose that $\Pi = \{\alpha_1, \ldots, \alpha_k\}$. Define ω_i by $\omega_i(h_{\alpha_j}) = \delta_{ij}$. Then $\{\omega_1, \ldots, \omega_k\}$ is a $\mathbb{Z}$-base of P. Show that $\rho = \sum_{i=1}^k \omega_i$.

5 (*optional*)**.** Show that if $\mathfrak{g} = \mathfrak{sl}(n)$, we have $P/Q \cong \mathbb{Z}/n\mathbb{Z}$.

6 (*optional*)**.** This question is about the root system of type G_2. Let $\epsilon_1, \epsilon_2, \epsilon_3$ denote the standard orthonormal basis of $\mathbb{R}^3$ with the usual Euclidean product $(\ ,\ )$. Let $\mathfrak{a}^*$ denote the subspace of $\mathbb{R}^3$ orthogonal to $\epsilon_1 + \epsilon_2 + \epsilon_3$. Denote

$$\Phi = \{\alpha \in \mathbb{Z}\langle \epsilon_1, \epsilon_2, \epsilon_2 \rangle \cap \mathfrak{a}^* \mid (\alpha, \alpha) = 2 \text{ or } 6\}.$$

(a) Verify that Φ is a root system and list the roots explicitly in coordinates.

(b) Choose as a base of simple roots $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3$. Draw the roots in the plane and indicate the fundamental Weyl chamber corresponding to this choice of simple roots.

(c) What is the order of the Weyl group W? How does W act on the roots?

(d) Show that $P = Q$.

7 (*optional*)**.** Suppose that we know that every $\mathfrak{g}$-module of length 2 is completely reducible. Prove that every $\mathfrak{g}$-module of finite length is completely reducible. (This is a general lemma that holds in every abelian category.)

[*Hint: prove by induction on the length of the module. If V is a reducible module of finite length, let S be a simple submodule of V, and consider $Q = V/S$.*]