C2.3 Representations of semisimple Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 2

1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k of characteristic 0. In lectures, we proved that the symmetrizing map

$$\phi : S(\mathfrak{g}) \to U(\mathfrak{g}),\ \phi(x_1 x_2 \dots x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(m)}$$

is a linear isomorphism.

(i) Let D be a derivation of $\mathfrak{g}$. Show that there exists one and only one derivation D' of $U(\mathfrak{g})$ which extends D.

(ii) If $D = \mathrm{ad}_{\mathfrak{g}}(x)$, show that $D'(u) = xu - ux$ for all $u \in U(\mathfrak{g})$.

(iii) Notice that there exists a unique derivation D'' of $S(\mathfrak{g})$ which extends D. Show that $D'' \circ \phi = \phi \circ D'$.

(iv) Recall that $\mathfrak{g}$ acts on both $S(\mathfrak{g})$ and $U(\mathfrak{g})$ via 'extending' the adjoint action of $\mathfrak{g}$ on itself to tensor products. Prove that ϕ is an isomorphism of $\mathfrak{g}$-modules.

(v) Deduce that there is a linear isomorphism between the spaces of $\mathfrak{g}$-invariants $S(\mathfrak{g})^{\mathfrak{g}} \cong U(\mathfrak{g})^{\mathfrak{g}}$. Check that the latter space is in fact the center of $U(\mathfrak{g})$. (Warning: this linear isomorphism is not an isomorphism of algebras, even though it is a linear isomorphism and both algebras are commutative!)

2. Let $\mathfrak{a}$ be a Lie subalgebra of the finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic 0.

(a) Using the PBW theorem show that the enveloping algebra $U(\mathfrak{a})$ is naturally a subalgebra of the enveloping algebra $U(\mathfrak{g})$.

(b) Let $\chi : U(\mathfrak{a}) \to \mathbb{C}$ be a homomorphism of (associative) $\mathbb{C}$-algebras such that $\chi(1) = 1$. Using the PBW theorem show that the vector space J spanned by $\{ua - \chi(a)u \mid u \in U(\mathfrak{g}),\ a \in U(\mathfrak{a})\}$ does not contain 1. Hence $U(\mathfrak{g})/J$ is nonzero. Check that $U(\mathfrak{g})/J$ has a natural structure of $\mathfrak{g}$-module.

(c) Let x, y be nonzero elements of $U(\mathfrak{g})$. Using the PBW theorem show that $x \cdot y$ is nonzero in $U(\mathfrak{g})$, meaning that $U(\mathfrak{g})$ has no zero divisors.

3. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie $\mathbb{C}$-algebra with a nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Let $\{x_i \mid 1 \le i \le n\}$ and $\{y_i \mid 1 \le i \le n\}$ be dual bases of $\mathfrak{g}$ with respect to B. Define the Casimir element of $\mathfrak{g}$ (with respect to B) to be $C = \sum_{i=1}^{n} x_i y_i$.

(i) Show that the definition of C does not depend on the choice of dual bases. Prove that C is in $Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$. [Hint: this is all in or around Lemma 15.7 in the Lie algebras notes for MT16, so you don't need to reproduce it here, but make sure that you know how to do this.]

(ii) Suppose that $\mathfrak{g}$ is a simple Lie algebra. Show that if C and C' are Casimir elements with respect to the forms B and B', respectively, than C' is a scalar multiple of C. [Hint: there is only one nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form up to scalar. Why?]

(iii) Suppose L is a finite dimensional simple $\mathfrak{g}$-module. Deduce that C acts on L by a scalar.

(iv) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and take $B(x, y) = \mathrm{tr}(xy)$, for $x, y \in \mathfrak{g}$. Verify that B is a nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form and define C with respect to convenient dual bases. How does C differ from C', the Casimir element defined with respect to the Killing form?

(v) Compute (in terms of λ) the scalar by which the Casimir element C from (iv) acts on a finite dimensional simple module $L(\lambda)$ with highest weight λ. [Hint: Compute the action of C on v, where v is a highest weight vector with weight λ, i.e.: $e \cdot v = 0$ for all $e \in \mathfrak{n}^+$ and $h \cdot v = \lambda(h)v$, for $h \in \mathfrak{h}$.]

4 (optional)**.** Let V be a finite dimensional complex vector space and $\mathfrak{g} = \mathfrak{sl}(V)$.

(i) Verify that $V \otimes V = S^2(V) \bigoplus \bigwedge^2 V$ is a decomposition as $\mathfrak{g}$-representations.

(ii) Show that $S^2(V)$ and $\bigwedge^2 V$ are simple $\mathfrak{g}$-modules.

(iii) Let $\mathfrak{g}' \subset \mathfrak{g}$ be an orthogonal (or symplectic) Lie algebra with respect to the nondegenerate bilinear symmetric (or skew-symmetric) form B. Then from (ii), $S^2(V)$ and $\bigwedge^2 V$ are $\mathfrak{g}'$-modules. Are they simple $\mathfrak{g}'$-modules?