C2.3 Representations of semisimple Lie algebras

Mathematical Institute, University of Oxford

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Problem Sheet 3

1. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie $\mathbb{C}$-algebra with a nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Let $\{x_i \mid 1 \leq i \leq n\}$ and $\{y_i \mid 1 \leq i \leq n\}$ be dual bases of $\mathfrak{g}$ with respect to B. Define the Casimir element of $\mathfrak{g}$ (with respect to B) to be $C = \sum_{i=1}^n x_i y_i$.

(i) Show that the definition of C does not depend on the choice of dual bases.

(ii) Prove that C is in $Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$. [Hint: read the proof of Lemma 15.7 in the Lie algebras notes for MT15.]

(iii) Suppose that $\mathfrak{g}$ is a simple Lie algebra. Show that if C and C' are Casimir elements with respect to the forms B and B', respectively, than C' is a scalar multiple of C. [Hint: there is only one nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form up to scalar. Why?]

(iv) Suppose L is a finite dimensional simple $\mathfrak{g}$-module. Show that C acts on L by a scalar and determine this scalar.

(v) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and take $B(x, y) = \mathrm{tr}(xy)$, for $x, y \in \mathfrak{g}$. Verify that B is a nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form and define C with respect to convenient dual bases. How does C differ from C', the Casimir element defined with respect to the Killing form?

(vi) Compute (in terms of λ) the scalar by which the Casimir element C from (v) acts on a Verma module $M(\lambda)$.

2. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and let $M(\lambda)$ be a Verma module. Here, we may think of λ as a complex number.

(i) Show from first principles, that if λ is a nonnegative integer then there exists an injective map $\phi : M(-\lambda - 2) \to M(\lambda)$ whose cokernel $L(\lambda) = M(\lambda)/\mathrm{Im}\ \phi$ is irreducible. How does this relate to the general theory that we've studied in the lectures?

(ii) Show that $M(\lambda) \otimes M(\mu)$ is not in category $\mathcal{O}$.

(iii) Consider $M(\lambda) \otimes L(\mu)$, where $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}_{\geq 0}$. We showed in the lectures that such a tensor product is still in $\mathcal{O}$, in particular, it has a finite composition series. Determine the composition series.

3. If $\lambda, \mu \in \mathfrak{h}^*$ and $w \in W$, verify that

$$w \cdot (\lambda + \mu) = w \cdot \lambda + \mu, \quad w \cdot \lambda - w \cdot \mu = w(\lambda - \mu).$$

Here $\cdot$ is the dot-action of W from the lectures.

4. Define $\tau : \mathfrak{g} \to \mathfrak{g}$ by $\tau(e_\alpha) = e_{-\alpha}$, $\alpha \in \Phi$ and $\tau(h) = h$ for all $h \in \mathfrak{h}$.

(i) Check that τ is an anti-involution of $\mathfrak{g}$, i.e., $\tau^2 = \mathrm{id}$ and $\tau([x, y]) = [\tau(y), \tau(x)]$, for all $x, y \in \mathfrak{g}$. We call τ the transpose map pf $\mathfrak{g}$. (This is motivated by the example $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.)

(ii) Extend τ to an anti-automorphism of $U(\mathfrak{g})$.

(iii) Prove that τ fixes $Z(\mathfrak{g})$ pointwise. [Hint: show that τ commutes with the Harish-Chandra projection.]

5. Let E be the strictly upper triangular matrix in $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ that has 1 in all entries $(i, i + 1)$ and 0 everywhere else. Let H be the diagonal matrix with entries $(n - 1, n - 3, \ldots, -(n - 1))$.

(i) Verify that $[H, E] = 2E$.

(ii) Determine a strictly lower triangular matrix F such that $\{E, H, F\}$ is a Lie triple. This is called the principal Lie triple of $\mathfrak{g}$. Notice also that this defines a Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$.

(iii) Determine the decomposition of $\mathfrak{g}$ into simple modules as an $\mathfrak{sl}(2) = \mathbb{C}\langle E, H, F\rangle$ module, under the adjoint action.

(iv) Let $\sigma = (n_1, n_2, \ldots, n_k)$ be a partition of n, i.e., $n_1 + n_2 + \cdots + n_k = n$ and $n_1 \geq n_2 \geq \cdots \geq n_k$. Let E_σ be the nilpotent matrix in the Jordan normal form corresponding to σ. By generalizing (ii), show that there exists H_σ and F_σ such that $\{E_\sigma, H_\sigma, F_\sigma\}$ is a Lie triple.

(v) Deduce that if E is any nilpotent element of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ then there exists a Lie triple in $\mathfrak{g}$ that contains E. (For an arbitrary semisimple Lie algebra, this is called the Jacobson-Morozov theorem.)