C2.3 Representations of semisimple Lie algebras

Mathematical Institute, University of Oxford
Hilary Term 2019

Problem Sheet 4

1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

(i) Let L be a finite dimensional $\mathfrak{g}$-module. Show that L is simple if and only if the dual module L^* is simple.

(ii) Let $L(\lambda)$, $\lambda \in P^+$ be a simple $\mathfrak{g}$-module with highest weight λ. Show that the dual $L(\lambda)^*$ is isomorphic to $L(-w_0(\lambda))$, where w_0 is the Weyl group element sending Φ^+ (the positive roots) to $-\Phi^+$.

(iii) What condition should λ satisfy such that 0 is a weight of $L(\lambda)$?

2. Use the Weyl dimensional formula to show that for every natural number k, there exists a simple $\mathfrak{g}$-module of dimension k^r, where r is the number of positive roots of $\mathfrak{g}$.

3. Let $\omega_1, \dots, \omega_n$ be the fundamental weights of the complex semisimple Lie algebra $\mathfrak{g}$. Show that every finite dimensional simple $\mathfrak{g}$-representation occurs as a direct summand in a suitable tensor product (repetitions allowed) of the simple modules $L(\omega_1), \dots, L(\omega_n)$. (We call these simple modules, the fundamental representations of $\mathfrak{g}$.)

4. Let $\mathfrak{g} = sl(n, \mathbb{C})$.

(i) Use Weyl's dimensional formula to show that $L(\omega_i) = \bigwedge^i V$, $1 \le i \le n-1$, where $V = \mathbb{C}^n$ is the standard representation.

(ii) Identify the adjoint representation in terms of the highest weight classification. (Why is the adjoint representation irreducible?)

5. Let $\mathfrak{g} = sl(3, \mathbb{C})$ and $L(\omega_1)$, $L(\omega_2)$ the two fundamental representations. Verify:

(i) $L(\omega_1)^* \cong L(\omega_2)$.

(ii) Konstant's multiplicity formula, and

(iii) Weyl's character formula for these two representations.

6. Let $\mathfrak{g} = sp(2n, \mathbb{C})$ realized as the space of matrices $X \in gl(2n, \mathbb{C})$ such that $X^t J + JX = 0$, where X^t is the transpose matrix, and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$; here I_n is the $n \times n$ identity matrix.

(i) Show that every $X \in \mathfrak{g}$ is of the form $X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$, where B and C are symmetric $n \times n$ matrices and A is an arbitrary $n \times n$ matrix.

(ii) Let $\mathfrak{h}$ be the subalgebra consisting of diagonal matrices. Determine the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ and the Cartan decomposition.

(iii) Choose the system of positive roots such that the corresponding root vectors lie in matrices of the form $\begin{pmatrix} A' & B \\ 0 & -A'^t \end{pmatrix}$, where A' is an upper triangular matrix and B is a symmetric matrix as before.

(iv) Determine the fundamental weights.

(v) Let $V = \mathbb{C}^{2n}$ be the standard representation of $\mathfrak{g}$ (which acts by matrix multiplication on column vectors). Show that V is an irreducible $\mathfrak{g}$-representation and it is in fact a fundamental representation.

(vi) Show that $\bigwedge^2 V$ decomposes as $W \bigoplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation and W is an irreducible (fundamental) representation.

(vii) For $sp(4, \mathbb{C})$, describe all the weights of the fundamental representations V and W and verify that the Weyl dimension formula holds.

(viii) In $sp(2n, \mathbb{C})$, show that the k-th fundamental representation is contained in $\bigwedge^k V$ and in fact it is precisely the kernel of the *contraction* map $\phi_k : \bigwedge^k V \to \bigwedge^{k-2} V$ defined by

$$\phi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k,$$

where Q is the skew-symmetric form defining $\mathfrak{g}$, i.e., $Q(v, u) = v^t J u$.

[*For this exercise, you may consult Section 16 in Fulton-Harris "Representation Theory", especially for the structural results on roots, Cartan decomposition etc.*]