Axiomatic Set Theory: Problem sheet 1

1. Which of the ZF axioms Extensionality, Empty Set, Power Set, Infinity hold in the structure $\langle \mathbb{Q}, < \rangle$? Does this change if you use the weak versions (i.e. the ones where $\leftrightarrow$ is replaced by $\rightarrow$)? Also, find an instance of Separation that is true in $\langle \mathbb{Q}, < \rangle$ and one that is false.

2. Assuming ZF^-, show that there exists a *transitive* set M such that
(a) $\varnothing \in M$, and
(b) if $x \in M$ and $y \in M$, then $\{x, y\} \in M$, and
(c) every element of M contains at most two elements.
Show further that if σ is an axiom of ZF^- other than Infinity, Union or Power Set, then $\langle M, \in \rangle \vDash \sigma$. (It follows that if ZF^- is consistent then so is ($\mathrm{ZF}^- \setminus \{$Infinity, Union, Power Set$\}$.)

3. Assuming ZF show that if a is a non-empty transitive set then $\varnothing \in a$.

4. Deduce Pairing from the other axioms of ZF^-.

5. Assuming ZF (ie. ZF^-+Foundation) prove that the following two definitions of "ordinal" are equivalent:
(i) An ordinal is a transitive set well-ordered by $\in$.
(ii) An ordinal is a transitive set totally ordered by $\in$.
Hence show that 'x is an ordinal' is absolute for non-empty transitive classes $A \subseteq B$ satisfying (enough of) ZF.

6. Check that if $A \subseteq B$ are non-empty transitive classes satisfying (enough of) ZF, $On \subseteq A$, F is a class function that is absolute for A, B and $a \in A$ is given by a defined notion absolute for A, B then the formula G given by the Recursion Theorem is absolute for A, B.

7. Read the 'Satisfaction' document to recall how we could define $(x, \in) \vDash \phi(x_1, \ldots, x_n, t)$ (in ZF) for a formula ϕ of LST.

8. This question seems long but should not be difficult (most should be one line, some have been done in the lecture notes). It is enough if you are convinced you could write down appropriate formulae. Let your class tutor know about any which you find tricky.

Give formulae of LST which express (are ZF-equivalent to) the following:

1. $x \subseteq y$
2. $z = \{x_1, \ldots, x_n\}$
3. $z = \langle x_1, \ldots, x_n \rangle$
4. x is an n-tuple
5. z is an n-tuple and $\pi_i(z) = x$
6. $z = x \cup y$
7. $z = x \cap y$
8. $z = \bigcup x$
9. $z = x \setminus y$
10. x is an n-ary relation on y
11. x is a function
12. $z = x \times y$
13. x is a function and $dom(x) = z$
14. x is a function and $z = ran(x)$
15. x is transitive
16. x is an ordinal
17. x is a successor ordinal
18. x is a limit ordinal
19. $x = \omega$

Which of these are (obviously) absolute for transitive non-empty classes $A \subseteq B$ satisfying (enough of) ZF?