Sheet 1

Question 1

Extensionality/A1 : Suppose $q, p \in \mathbb{Q}$ with $q \neq p$, wlog $q < p$. Then there is $r \in (q, p) \cap \mathbb{Q}$ anf $r < p$ and $\neg(r < q)$. Hence $\mathbb{Q} \models$ **A1**

A2 : If $q \in \mathbb{Q}$ then $q - 1 \in \mathbb{Q}$ and $q - 1 < q$. Thus $\neg\mathbb{Q} \models$ **A2**. Note that this implies that $\mathbb{Q}$ does not satisfy **Separation** either since $\mathbb{Q}$ is non-empty.

Powerset/A7: First note that $r \subseteq q$ means $\forall t\ (t < r \to t < q)$, i.e. $r \leq q$.

Here is a subtlety: if you take the **Powerset** from the lectures:

$$\forall x \exists z \forall r\ \ [r \in z \to r \subseteq x]$$

then $\mathbb{Q} \models$ **Powerset**: Let $q \in \mathbb{Q}$. Try $z = q$: if $r \in \mathbb{Q}$ such that $r < z = q$ then certainly $r \leq q$.

However if you take **A7**:

$$\forall x \exists z \forall r\ \ [t \in z \leftrightarrow r \subseteq x]$$

then $\mathbb{Q} \not\models$ **A7**: take $q = 0$ and any $z \in \mathbb{Q}$: if $q < z$ then take $r \in \mathbb{Q}$ with $q < r < z$ (e.g. $r = (q + z)/2$ and note that $r \not\leq q$ but $r < z$. If $z \leq q$ then take $r = q$ and note that $r \leq q$ but $r \not< z$.

So in the absence of **Separation** (see below) the distinction becomes important.

A8: As stated, $\neg\mathbb{Q} \models A8$ since there is no $y \in \mathbb{Q}$ such that $\forall z\ z \notin y$.

Separation/A5: Let $\phi(r) \equiv r < r$ and fix $q \in \mathbb{Q}$. We ask whether there is any $p \in \mathbb{Q}$ such that $\forall t\ \ [t < p \leftrightarrow t < q \wedge \phi(t)]$? Suppose there was some such p and consider $t = \min\{p - 1, q - 1\} \in \mathbb{Q}$. Then $t < p$ but $\neg\phi(t)$ a contradiction.

Let $\phi(r, q) \equiv r < q$. Fix $q, s \in \mathbb{Q}$. We ask whether there is any $p \in \mathbb{Q}$ such that $\forall t\ \ [t < p \leftrightarrow t < s \wedge \phi(t, q)]$. Clearly $p = \min\{s, q\}$ satisfies this.

Question 2

By recursion on ω: we need to show that $P(x) = x \cup \{\{y, z\} : y, z \in x\}$ is a set (see below) and use $\{\langle x, P(x)\rangle : x \in U\}$ as our class function F to obtain M_n for $n \in \omega$ such that $M_0 = \{\emptyset\}$ and $M_{n+1} = P(M_n)$. We finally use **Replacement**, **Infinity** (to get that ω is a set) and **Union** to define $M = \bigcup\{M_n : n \in \omega\}$ as a set.

First we show by induction on n that no element of M_n contains more than two elements (straightforward) and deduce that no element of M contains more than two elements. For transitivity assume $x \in M$. Find the least n such that $x \in M_n$. If $n = 0$ then $x = \emptyset$ and we are vacuously done. Otherwise $n = m + 1$

for some m and there are $y, z \in M_n$ with $x = \{y, z\}$. So now assume $t \in x$. Then $t = y$ or $t = z$. In either case $t \in M_n \subseteq M$ as required.

Note that clearly $M_n \subseteq M_{n+1}$ by construction and hence by induction we have $n \leq m$ implies $M_n \subseteq M_m$. Thus if $x, y \in M$ then $\{x, y\} \in M$ (for a more formal proof see below).

For the axioms: **A1/Extensionality** follows from transitivity of M, **A2/Emptyset** is trivial. For **A3/Pairing** follows by construction: let $x, y \in M$ and find n, m such that $x \in M_n, y \in M_m$. Wlog $n \leq m$ and by the note above we then have $x, y \in M_m$ so that $\{x, y\} \in M_{m+1} \subseteq M$.

For **Separation**, let $\phi(y; v_1, \ldots, v_n)$ be a formula, $a_1, \ldots, a_n \in M$ and $u \in M$. Let n be least such that $u \in M_n$. If $n = 0$ then $u = \emptyset$, so let $z = \emptyset \in M$ and $M \models t \in z \leftrightarrow t \in u \wedge \phi(t; a_1, \ldots, a_n)$ s vacuously true. So assume $n = m + 1$ for some m. By leastness, there are $x, y \in M_m$ with $u = \{x, y\}$. Set $z = \left\{t \in u : \phi(t; a_1, \ldots, a_n)^M\right\}$. Then z is one of $\emptyset$, $\{x\}$, $\{y\}$ or $\{x, y\}$ all of which belong to M. Finally $M \models t \in z \leftrightarrow t \in u \wedge \phi(t; a_1, \ldots, a_n)^M$ as required.

For **Replacement** assume that $\phi(x, y)$ is a formula and

$$\forall x, y, y' \in M \left[\phi(x, y)^M \wedge \phi(x, y')^M \rightarrow y = y'\right].$$

Let $s \in M$ and define $z = \left\{y \in M : \exists x \in M(x \in s \wedge \phi(x, y)^M)\right\}$. Firstly, z is a set by **Separation**. If we can show that $z \in M$ then we are done. So, let n be least such that $s \in M_n$. If $n = 0$ then $s = \emptyset$ and hence $z = \emptyset$ and clearly $M \models \forall t \ [t \in z \leftrightarrow t \in s \wedge \ldots]$. Otherwise let $n = m + 1$ and by leastness find $u, v \in M_m$ with $s = \{u, v\}$. Then there are is at most one $a \in M$ with $\phi(u, a)^M$ and only one $b \in M$ with $\phi(v, b)^M$. But all of $\emptyset, \{a\}, \{b\}, \{a, b\}$ (depending on whether or not $a, b \in M$ exist) belong to M. So $z \in M$ as required.

Finally **A10**: There are of course multiple versions of the Choice axiom (which are equivalent under ZF). We will look at two of them:

Question 3

Let a be non-empty, transitive and let m be its $\in$-minimal element (from Foundation). If $x \in m$ then by transitivity $x \in a$, contradicting minimality of m. So $m = \emptyset$ as required.

Question 4

Suppose x, y are sets. Write $0 = \emptyset$ and $1 = \{\emptyset\} = \mathcal{P}(\emptyset)$. Then $0, 1 \subseteq 1$ so $0, 1 \in \mathcal{P}(1)$ so by **Separation** $\{0, 1\}$ is a set (in fact, $\mathcal{P}(1) = \{0, 1\}$ so another application of **Powerset** can avoide **Separation**). By Replacement (with $\phi(r, t)$ as

$$(r = 0 \wedge t = x) \vee (r = 1 \wedge t = y) \vee t = \emptyset$$

this gives that $\{x, y\}$ is a set.

Note that we are using a very weak form of **Replacement** here.

Question 5

Clearly being well-ordered implies being totally ordered so (i) implies (ii). We focus on (ii) implies (i): Suppose that α is transitive and totally ordered by $\in$. Let $x \subseteq \alpha$ and assume that $x \neq \emptyset$. Apply **Foundation** to find $m \in x$ such that $m \cap x = \emptyset$. Since α is transitive, $m \in \alpha$ and by construction m is the $\in$-minimal element of x.

For the deduce, note that α is transitive and totally ordered by $\in$ is a Δ_0 formula, so absolute for transitive non-empty classes $A \subseteq B$. As long as A, B satisfy **Foundation**, the above show that $A \models \alpha$ is transitive and well-ordered by $\in$ if and only if $A \models \alpha$ is transitive and totally-ordered by $\in$ if and only if $B \models \alpha$ is transitive and totally-ordered by $\in$ if and only if $B \models \alpha$ is transitive and well-ordered by $\in$, as required.

Question 6

The most difficult part is to find out what you are actually asked to do. We want to show that: If

- A, B satisfy enough of ZF$^-$ so that the Recursion Theorem on On holds and
- $a \in A$ and
- that F is a formula such that $A \models F$ is a class function (we will write $F^A(a)$ for the unique $y \in A$ with $A \models F(a, y)$) and
- $B \models F$ is a class function (similarly $F^B(b)$ is the unique $y \in B$ such that $B \models F(a, b)$) and
- F is absolute for A, B (i.e. for $a \in A$, $F^A(a) = F^B(a)$) and
- G_A (resp. G_B) are formulae such that

$$A \models G \text{ is a class function on } On^A \wedge \tag{1}$$

$$G(0^A) = a \tag{2}$$

$$\wedge \forall \alpha \in \mathrm{On}^A \; G_A(\alpha + 1) = F^A(G_A(\alpha)) \tag{3}$$

$$\wedge \forall \gamma \in \mathrm{Lim}^A \; G_A(\gamma) = \left(\bigcup \{G_A(\beta) : \beta \in \alpha\}^A \right)^A \tag{4}$$

  (resp. the above for B and G_B) where all the superscript As mean that we should interpret this formula in A

then

$$\forall \alpha \in On^A \; G_A(\alpha) = G_B(\alpha).$$

For the proof, we first note that since A, B are non-empty transitive classes satisfying enough of ZF we have $\emptyset^A = \emptyset^B$, $On^A \subseteq On^B$ and $Lim^A \subseteq Lim^B$

(being an ordinal is absolute and being a successor ordinal is absolute, hence being a limit ordinal is absolute).

So assume there is $\alpha \in On^A$ with $G_A(\alpha) \neq G_B(\alpha)$. Since A satisfies enough of ZF, there is a minimal such α, say α_0.

Case $\alpha_0 = 0^A = 0^B$: Then $G_A(\alpha_0) = a = B_B(\alpha_0)$, a contradiction.

Case α_0 is a successor (in A): Being a successor is absolute for A, B, so α_0 is successor in B. Let $\beta_A \in On^A$ be such that $A \models \alpha_0 = \beta_A + 1$ and similarly for β_B. Since **Pairing** and **Union** are absolute, $A, B \models \beta_A + 1 = \beta_B + 1$ and it follows that $A, B \models \beta_A = \beta_B$. We will simply write β for β_A. Since $\beta \in \alpha_0$, by minimality of α_0 we must have

$$G_A(\alpha_0) = F^A(G_A(\beta)) = F^A(G_B(\beta) = F^B(G_B(\beta)) = G_B(\alpha_0)$$

(where the second $=$ comes from the minimality of α_0 and the third from absoluteness of F), giving another contradiction.

Case α_0 is a limit (in A): Again, α_0 will be a limit in B. Now apply minimality of α_0 to see that for $\beta in \alpha_0$, $G_A(\beta) = G_B(\beta)$, so that $\{G_A(\beta) : \beta \in \alpha\}^A = \{G_B(\beta) : \beta in \alpha\}^B$, so that by absoluteness of $\bigcup$, we get $G_A(\alpha_0) = G_B(\alpha_0)$.

Question 7

An expanded solution of this (containing almost all details, I hope) can be found in the 'Satisfaction' file.

Summary In the finitistic meta-theory, we can define (using a Gödel numbering scheme) an injection $\lceil . \rceil$: Formulae of LST $\to \omega$ together with a function $free : \omega \to$ finite subsets of ω such that $free(\lceil \phi \rceil)$ is the set of indices of free variables of ϕ such that these definitions only use basic arithmetic on ω.

By recursion on ω we can then define for each set $x \in V$, a $0,1$-valued function (in our universe V satisfying enough of ZF) $val_x : \omega \times \bigcup \left\{ x^d : d \text{ finite } \subseteq \omega \right\} \to \{0,1\}$ such that for a formula $\phi(v_{i_1}, \ldots, v_{i_n})$ with all free variables displayed, and $v \in \bigcup \left\{ x^d : d \text{ finite } \subseteq \omega \right\}$ with $free(\lceil \phi \rceil) \subseteq d$ we have (in the meta-theory) $val_x(\lceil \phi \rceil, v) = 1$ if and only if $x \models \phi(v(i_1), \ldots, v(i_n))$.

Moreover, val_x is absolute for transitive non-empty classes A, B satisfying enough of ZF (in the sense that $val_x^A = val_x^B$ for $x \in A$).

Some details: Assume you have a Gödel numbering and the function free as described above. When mentioning formulae ϕ or free variables below, these should of course be replaced by the Gödel number or a suitable statement involving $free$ (and the Gödel number).

We of course need the concept of an assignment: an assignment for $\phi(v_{i_1}, \ldots, v_{i_n})$ (with free variables shown) is a function $v : m \to x$ where $m \supseteq \{i_1, \ldots, i_n\}$ and

m is a finite subset of ω. We define $val_x(\ulcorner\phi\urcorner, v)$, also written as $(x, \in) \models_v \phi(v_1, \ldots, v_n)$, by recursion on the complexity of ϕ:

- "$(x, \in) \models_v v_{i_1} = v_{i_2}$" $= 1$ if and only if v is an assignment for ϕ (i.e. $i_1, i_2 \in free(\ulcorner\phi\urcorner)$) and $v(i_1) = v(i_2)$;
- "$(x, \in) \models_v v_{i_1} \in v_{i_2}$" $= 1$ if and only if v is an assignment for ϕ and $v(i_1) \in v(i_2)$;
- "$(x, \in) \models_v \psi_1 \wedge \psi_2$" $= 1$ if and only if v is an assignment for ϕ and "$(x, \in) \models_v \psi_1$" $= 1$ and "$(x, \in) \models_v \psi_2$" $= 1$;
- "$(x, \in) \models_v \neg\psi$" $= 1$ if and only if v is an assignment for ϕ and "$(x, \in) \models_v \psi$" $= 0$;
- "$(x, \in) \models_v \exists v_{i_k} \psi(v_{i_1}, \ldots, v_{i_n}, v_{i_{n+1}})$" $= 1$ if and only if v is an assignment for ϕ and for some assignment v' for ψ such that $dom(v') = dom(v) \cup \{i_k\}$ and $\forall i \in dom(v) \setminus \{i_k\}\, v(i) = v'(i)$ (this codes the idea that v' is an extension for v, possibly redefining the value of the 'dummy' variable "v_{i_k}) "$(x, \in) \models_{v'} \psi(v_{i_1}, \ldots, v_{i_{n+1}})$" $= 1$;

(we replace $\vee$, $\forall$ by logically equivalent formulae involving only the symbols used above).

This can be done in basic set theory: we only need

- being a finite ordinal, basic arithmetic on finite ordinals;
- the set of functions with domain a finite set of ordinals and range x (the set of potential valuations);
- the recursion theorem on finite ordinals (i.e. on the class ω);

Instead of recursing on the complexity of ϕ, we recurse on a measure of the complexity of ϕ coded by finite ordinals. Details of (one way of) how to code this formally, can be found in the relevant chapter of the lecture notes, but are not examinable.

Note that all of the above are absolute (and exist) for classes that satisfy (enough of) $\mathrm{ZF} - P$ and since (class) functions defined by recursion on absolute notions are absolute, we get that val_x is absolute for transitive, non-empty classes $A \subseteq B$ satisfying (enough of) $\mathrm{ZF} - P$ and $x \in A$.

Question 8

We need to check that the well-orders defined in the lecture notes are indeed well-orders. None of these checks are difficult.

Question 9

1. $x \subseteq y \equiv \forall t \in x\,[t \in y]$ which is Δ_0 so absolute.

2. $z = \{x_1, \dots, x_n\} \equiv x_1 \in z \wedge \dots x_n \in z \wedge \forall t \in z\,[t = x_1 \vee \dots \vee t = x_n]$ which is Δ_0 so absolute.

3. $z = \langle x_1, \dots, x_n\rangle$: We define this by induction (in the meta-theory) as follows:

$$z = \langle\rangle \equiv z = \emptyset \equiv \forall t \in z\,[t \neq t] \tag{5}$$
$$z = \langle x_1\rangle \equiv z = \{x_1\} \tag{6}$$
$$z = \langle x_2\rangle \equiv z = \{\{x_1\}, \{x_1, x_2\}\} \tag{7}$$
$$z = \langle x_1, x_2, \dots, x_{n+1}\rangle \equiv z = \langle\langle x_1, \dots, x_n\rangle, x_{n+1}\rangle \tag{8}$$

   We could of course write out a formula for each n, but this would be painful. However, all the 'defined' notions which we use are Δ_0 so the formula we would to write down (if we were forced to do so) are Δ_0.

   The alternative is to define the two-tuple, some totally ordered set of size n (e.g. $n \in \omega$) and then $z = \langle x_1, \dots, x_n\rangle$ by $z = \{\langle 0, x_1\rangle, \dots, \langle n-1, x_n\rangle\}$.

4. x is an n-tuple: The obvious definition $\exists x_1, \dots, x_n z = \langle x_1, \dots, x_n\rangle$ is not Δ_0. But we can be slightly tricky as follows:

$$\exists x_n, t_{n-1} \in z \exists x_{n-1}, t_{n-2} \in t_{n-1} \dots \exists x_2, t_1 \in t_2 \exists x_1 \in t_1\,[z = \langle x_1, \dots, x_n\rangle] \tag{9}$$

   and this is Δ_0.

   So (the important case), z is a two-tuple would be

$$\exists x_2, t_1 \in z \exists x_1 \in t_1\,[z = \langle x_1, x_2\rangle]\,. \tag{10}$$

   Similarly, if you define the tuple via functios, you can be crafty and write down a Δ_0 formula for a given n.

5. z is an n-tuple and $\pi_i(z) = x$: We write down the formula above but also but in $\wedge x_i = x$ and again we have absoluteness. Explicitly:

$$\exists x_n, t_{n-1} \in z \exists x_{n-1}, t_{n-2} \in t_{n-1} \dots \exists x_2, t_1 \in t_2 \exists x_1 \in t_1\,[z = \langle x_1, \dots, x_n\rangle \wedge x_i = x] \tag{11}$$

   Or we could do this inductively, saying

$$z \text{ is a 0-tuple } \equiv z = \emptyset \tag{12}$$

6. $z = x \cup y$: Either we define this as $z = \bigcup \{x, y\}$ (for $\bigcup$ see later - but this only makes sense in the presence of **Pairing**) or explicitly as

$$\forall t \in z\,[\exists w \in x\,[t \in w] \vee \exists w \in y\,[t \in w]] \tag{13}$$
$$\wedge \forall t \in x\,[t \in z] \wedge \forall t \in y\,[tinz] \tag{14}$$

   which is Δ_0 so absolute.

7. $z = x \cap y$: We could use separation, but it is less demanding to define it as

$$\forall t \in z\,[t \in x \wedge t \in y] \tag{15}$$
$$\wedge \forall t \in x\,[t \in y \to tinz] \tag{16}$$

which is again Δ_0.

8. $z = \bigcup x$: Instead of the 'obvious'$\forall t\,[t \in z \leftrightarrow \exists y \in x\,[t \in y]]$ which is not Δ_0, we can use

$$\forall t \in z \exists y \in x\,[t \in y] \tag{17}$$
$$\wedge \forall y \in x \forall t \in y\,[t \in z] \tag{18}$$

which is Δ_0.

9. $z = x \setminus y$:

$$\forall t \in z\,[t \in x \wedge \neg\,[t \in y]] \tag{19}$$
$$\wedge \forall t \in x\,[\neg\,[t \in y] \to t \in z] \tag{20}$$

is Δ_0.

10. x is an n-ary relation on $y_1, \ldots, y_n$ (take all the y_i equal to y):

$$\forall t \in x \exists x_1 \in y_1, \ldots x_n \in y_n\,[t = \langle x_1, \ldots, x_n \rangle] \tag{21}$$

is Δ_0.

11. x is a function:

$$\forall t \in x\,[t \text{ is a 2-tuple}] \tag{22}$$
$$\wedge \forall t_1, t_2 \in x\,[\pi_1(t_1) = \pi_1(t_2) \to t_1 = t_2] \tag{23}$$

where $\pi_1(t_1) = \pi_1(t_2)$ should of course be replaced by the appropriate formula from above, namely

$$\exists w \in t_1 \exists u \in t_2 \exists x_1, x_2 \in w \exists y_1, y_2 \in u\,[t_1 = \langle x_1, x_2 \rangle \wedge t_2 = \langle y_1, y_2 \rangle \wedge x_1 = y_1] \tag{24}$$

and everything is Δ_0

12. $z = x \times y$:

$$\forall t \in z \exists x_1 \in x \exists y_1 \in y\,[t = \langle x_1, y_1 \rangle] \tag{25}$$
$$\wedge \forall x_1 \in x \forall y_1 \in \exists t \in z\,[t = \langle x_1, y_1 \rangle] \tag{26}$$

is Δ_0

13. x is a function and $dom(x) = z$:

$$x \text{ is a function} \tag{27}$$
$$\wedge \forall t \in x \; \pi_1(t) \in z \tag{28}$$
$$\wedge \forall w \in z \exists t \in x \; \pi_1(t) = w \tag{29}$$

where $\pi_1(t) \in z$ should of course be replaced by an appropriate Δ_0 formula.

14. x is a function and $ran(x) = z$: very similar to the previous one.

15. x is transitive:

$$\forall y \in x \forall t \in y \, [t \in x] \tag{30}$$

is Δ_0

16. x is an ordinal: This one is not absolute for transitive classes satisfying only ZF$^-$! See the lecture notes. However, assuming foundation, there is an equivalent definition which is absolute.

17. x is a successor ordinal:

$$x \in On \wedge \exists y \in x \, [x = y \cup \{y\}] \tag{31}$$

and this is absolute provided being and ordinal is absolute.

18. x is a limit ordinal: either x is an ordinal and not a successor ordinal or x is an ordinal and $\forall y \in x \exists z \in x \, [z = y \cup \{y\}]$. Again, this is absolute provided being an ordinal is absolute.

19. $x = \omega$:

$$x \text{ is a limit ordinal} \tag{32}$$
$$\wedge \forall y \in x \, [y \text{ is a successor ordinal} \vee y = \emptyset] \tag{33}$$

which is absolute if being an ordinal is absolute.

20. x is a finite subset of z:

$$\exists n \in \omega \exists f \;\; [f \text{ is a function} \wedge dom(f) = n \wedge ran(f) = x] \tag{34}$$

Note that this is not obviously absolute, since the $\exists f$ quantifier is not bounded.

However, for transitive classes A, B satisfying (enough of) ZF $- P$ we can show the following: if

$$A \models x \text{ is a finite subset of } z$$

then so does B since the witnessing function f in A also belongs to B (and the things we assert about f are absolute). So suppose now that

$$B \models x \text{ is a finite subset of } z$$

Let $n \in \omega^B = \omega^A$ (by absoluteness of ω) and $f \in B$ witness this. Then $n \in A$ and if we can show $f \in A$ then we are done (by the absoluteness of what we assert about f and n).

First we show that if $a, b \in A$ or B and then $a \times b \in A$ (or B): we note that

$$a \times b = \bigcup_{y \in b} a \times \{y\} = \bigcup_{y \in b} \bigcup_{x \in a} \{\langle a, y \rangle\}$$

and for each $y \in b$, $x \in a$, $\langle x, y \rangle \in A$ (by **Pairing**), so for each $y \in b$, $\{\langle x, y \rangle : x \in a\} \in A$ (by **Replacement**), so for each $y \in b$,

$$a \times \{y\} = \bigcup \{\langle x, y \rangle : x \in a\} \in A$$

(by **Union**) and another application of **Replacement** and **Union** gives $a \times b \in A$.

Next if $a, b \in A$ then $a \cap b \in A$: this is a simple application of **Separation**, observing $a \cap b = \{t \in a : t \in b\}$.

Next we show by induction on $m \in \omega$ that if $f \in B$ and f is a function on m with $ran(f) \subseteq z$ then $f \in A$: for $m = 0$ we must have $f = \emptyset \in A$ (by **Emptyset** or non-emptyness of A, **Separation** and absoluteness of $\emptyset$). Assume this is true for m and let $f \in B$ be a function on $m + 1$. Then $f|_m = f \cap m \times z \in B$ so by inductive assumption $f|_m \in A$. Also, by **Pairing** and **Infinity** and transitivity of A $\{\langle m, f(m) \rangle\} \in A$ so that $f = f|_m \cup \{\langle m, f(m) \rangle\} \in A$ as required.

21. $z =$ the set of finite subsets of x:

$$\forall t \in z \, [t \text{ is a finite subset of } x] \tag{35}$$
$$\wedge \forall y \, [y \text{ is a finite subset of } x \to y \in t] \, . \tag{36}$$

The first line is absolute for non-empty transitive classes A, B satisfying enough of $\mathrm{ZF} - P$. In the second line, we need to show that if $y \in B$ and y is a finite subset of x then $y \in A$ (then absoluteness kicks in). If $y \in B$ and y is a finite subset of x then we can find a witnessing $n \in \omega, f : n \to x$ with $ran(f) = y$ with $f \in B$. Thus $f \in A$ by the above. Now we need to show that if $f \in A$ then $ran(f) \in A$. But this follows from **Replacement**:

$$ran(f) = \{y : \exists m \in n \; f(m) = y\}$$