Sheet 2

Question 1

See lecture notes (complete proofs are given).

Question 2

We define $\alpha^0 = 1$, $\alpha^{\beta+1} = \alpha^\beta.\alpha$ and for $\gamma \in \text{Lim}$, $\alpha^\gamma = \bigcup_{\beta<\gamma} \alpha^\beta$. By induction on On we easily see that $\alpha^\beta \in \text{On}$ provided $\alpha, \beta \in \text{On}$.

We need some facts about ordinal addition and multiplication:

- if x is a set of ordinals then $\sup x = \bigcup_{\alpha \in x} \alpha$;
- if x is an unbounded set in $\gamma \in \text{Lim}$ then $\sup x = \gamma$;
- thus for γ a limit ordinal and x unbounded in γ we have $\alpha + \gamma = \bigcup_{\beta \in x} \alpha + \beta$, $\alpha.\gamma = \bigcup_{\beta \in x} \alpha.\beta$ and $\alpha^\gamma = \bigcup_{\beta \in x} \alpha^\beta$;
- ordinal addition and multiplication are non-decreasing in their second variable;
- ordinal addition and multiplication are associative;
- if $\gamma \in \text{Lim}$ and $\beta \in \text{On}$ then $\beta + \gamma, \beta.\gamma, \beta^\gamma \in \text{Lim}$;

In the following, we always assume that α is a non-zero ordinal:

First we show that α^β is non-decreasing in β: it is enough to show $\alpha^\beta \leq \alpha^{\beta+1}$ as then the claim follows by induction. If $\alpha = 0$ this is trivial. Otherwise $\alpha^{\beta+1} = \alpha^\beta.\alpha \geq \alpha^\beta$ as multiplication is non-decreasing and $\alpha \geq 1$.

Next we show $\alpha^{\beta+\gamma} = \alpha^\beta.\alpha^\gamma$: we induct on γ. The base case is trivial. For the successor case

$$\alpha^{\beta+(\gamma+1)} = \alpha^{(\beta+\gamma)+1} = \alpha^{\beta+\gamma}.\alpha = \alpha^\beta.\alpha^\gamma.\alpha = \alpha^\beta.\alpha^{\gamma+1}$$

where the first equality is the (recursive) definition of ordinal addition, the second fact the (recursive) definition of exponentiation, the third the inductive hypothesis and the last the (recursive) definition (together with various omitted facts about ordinal multiplication).

For the limit case:

$$\begin{aligned}
\alpha^{\beta+\gamma} &= \bigcup_{\delta<\beta+\gamma} \alpha^{\delta} \quad \text{as } \beta+\gamma \text{ is a limit} \\
&= \bigcup_{\delta<\gamma} \alpha^{\beta+\delta} \quad \text{as } \{\beta+\delta : \delta<\gamma\} \text{ is unbounded in } \beta+\gamma \\
&= \bigcup_{\delta<\gamma} \alpha^{\beta}.\alpha^{\delta} \quad \text{inductive hypothesis} \\
&= \alpha^{\beta}\alpha^{\gamma}
\end{aligned}$$

where the last inequality follows from the fact that α^{γ} is a limit if γ is and that $\left\{\alpha^{\delta} : \delta < \gamma\right\}$ is unbounded in γ.

Now we do $\alpha^{\beta\cdot\gamma} = (\alpha^{\beta})^{\gamma}$: Again, we induct on γ. The base case is trivial. For the successor step

$$\alpha^{\beta.(\gamma+1)} = \alpha^{\beta\cdot\gamma+\beta} = \alpha^{\beta\cdot\gamma}.\alpha^{\beta} = (\alpha^{\beta})^{\gamma}.\alpha^{\beta} = (\alpha^{\beta})^{\gamma+1}$$

For the limit case:

$$\alpha^{\beta\cdot\gamma} = \bigcup_{\delta<\beta.\gamma} \alpha^{\delta} = \bigcup_{\delta<\gamma} \alpha^{\beta.\delta} = \bigcup_{\delta<\gamma} (\alpha^{\beta})^{\delta} = (\alpha^{\beta})^{\gamma}$$

where each equality is either from the definition or justified as for $\alpha^{\beta+\gamma} = \alpha^{\beta}\alpha^{\gamma}$.

Inductively, $2^n < \omega$ for $n \in \omega$: $2^{n+1} = 2^n.2 = 2^n + 2^n$. Now consider the case that $n = 0$ so that $2^n + 2^n = 2 = (0+1)+1 < \omega$ and $n = m+1$ giving $2^n + 2^{m+1} = (2^n + 2^m) + 1 + 1 < \omega$. Thus $2^{\omega} \leq \omega$.

But if $n \in \omega$, $n \geq 1$ then again by induction $2^n \geq n$ so that $\{2^n : n \in \omega\}$ is unbounded in ω and hence $2^{\omega} \geq \omega$.

Question 3

Define by recursion on $n \in \omega$, $\alpha_0 = \alpha + 1$ and $\alpha_{n+1} = F(\alpha_0)$ and let $\gamma = \bigcup_{n\in\omega} \alpha_{n+1}$ (this is the union of a set of ordinals, hence an ordinal). We claim that $F(\gamma) = \gamma$. If $F(\alpha_0) = \alpha_0$ then inductively $\alpha_n = \alpha_0$ for all $n \in \omega$ and hence $\gamma = \alpha_0 = F(\alpha_0) = F(\gamma)$. Otherwise $\alpha_0 < \alpha_1$ and by induction on n we see that $\alpha_n < \alpha_{n+1}$ for all n so that $\gamma \in \mathrm{Lim}$. Then $F(\gamma) = \bigcup_{\beta<\gamma} F(\beta)$. Now, if $\eta \in \gamma$ then $\beta \in \alpha_n \subseteq \alpha_{n+1} = F(\alpha_n) \in F(\gamma)$.

Conversely, if $\eta \in F(\beta)$ for $\beta < \gamma$ then $\beta \in \gamma$ so there is α_n with $\beta < \alpha_n$ and hence $\eta \in F(\beta) \subseteq F(\alpha_n) = \alpha_{n+1} \subseteq \gamma$.

The smallest non-zero fixed point of $F(x) = \omega.x$ is ω^ω: first $\omega.\omega^\omega = \sup_{n\in\omega} \omega.\omega^n = \sup_{n\in\omega} \omega^{n+1} = \omega^\omega$; secondly, if $\alpha < \omega^\omega$ then $\alpha < \omega^n$ for some n. Let n be least such that $\alpha < \omega^n$. Either $n = 0$ giving $\alpha = 0$ which was disallowed. Or $n = m + 1$ and then $\omega^m \leq \alpha < \omega^{m+1}$. But then (as multiplication is non-decreasing) $F(\omega^m) = \omega.\omega^m = \omega^{m+1} = \omega^n > \alpha = F(\alpha)$, a contradiction.

The Transitive Closure

Given a set x we want to define the transitive closure of x, denoted by $TC(x)$, as the smallest transitive set containing x as a subset. To do so, we have to construct a transitive set containing x: By recursion on ω we define $x_0 = x$ and $x_{n+1} = x_n \cup \bigcup x_n$. The x_n are sets by **Union** and **Pairing** and induction. Also $x_n \subseteq x_{n+1}$ for each $n \in \omega$ and hence $\forall n, m \in \omega\ [n \leq m \to x_n \subseteq x_m]$ (by induction on $m \geq n$). By **Replacement** and **Union** we then have that $z = \bigcup_{n\in\omega} x_n$ is a set.

We now claim that z is transitive and contains x as a subset: since $x_0 = x$ the latter is trivial. For the former, assume $u \in w \in z$. Find $n \in \omega$ such that $w \in x_n$ and then note that $u \in \{t : \exists y\ t \in y \in x_n\} = \bigcup x_n \subseteq x_{n+1}$ so that $u \in z$ as required.

We could now either apply **Separation** to form the smallest transitive subset containing x as a subset. Or we show that z is as required: for suppose z' is transitive and contains x as a subset. Then $x_0 = x \subseteq z'$ and by transitivity $\forall w\ w \subseteq z' \to \bigcup w \subseteq z'$ so that inductive each of the $x_n \subseteq z'$ and hence $z \subseteq z'$ as required.

Let us also note that $TC(\{x\})$ is the smallest transitive set containing x as an element.

Question 4

Suppose $H_\omega \neq V_\omega$. Assume first that $H_\omega \setminus V_\omega \neq \emptyset$ and pick $x \in H_\omega \setminus V_\omega$. If $x \subseteq V_\omega$ then for each $t \in x$, let $n_t \in \omega$ be least with $t \in V_{n_t}$. As x is finite, $N = \max_t n_t \in \omega$ exists and then $x \subseteq V_N$, giving $x \in \mathcal{P}(V_N) = V_{N+1} \subseteq V_\omega$, a contradiction. Thus $TC(x) \setminus V_\omega \supseteq x \setminus V_\omega \neq \emptyset$ and thus $TC(x) \setminus V_\omega$ (it is a set by **Separation**) has a $\in$-minimal element m. As $\emptyset \in V_\omega$, $m \neq \emptyset$. By assumption $m \in H_\omega$ (using $m \in TC(x) \to TC(m) \subseteq TC(x)$) and by minimality and transitivity of $TC(x)$, $\forall t \in m\ t \in V_\omega$. But as above, this gives $m \subseteq V_N$ for some $N \in \omega$ and thus $m \in V_{N+1}$, a contradiction.

Next we show by induction on n, that V_n is finite (and in fact has size 2^n).

Next, assume that $V_\omega \setminus H_\omega \neq \emptyset$ and we again pick a $\in$-minimal element $m \in V_\omega \setminus H_\omega$ (we already know that V_ω is transitive and that $V_\omega \setminus H_\omega$ is a set). If $m = \emptyset$ then we are done as $TC(\emptyset) = \emptyset$. Otherwise $m \in V_{n+1}$ for some n and thus $m \subseteq V_n$ and m is finite. For $t \in m$ we have $t \in V_\omega$ by transitivity of V_ω and $t \in H_\omega$ by minimality of m. Thus $TC(m) = \{m\} \cup \bigcup_{t \in m} TC(t)$ is a finite union of finite sets, so finite. Hence $m \in H_\omega$, a contradiction.

Question 5

Only the forward direction is interesting (since $V \models$ **Foundation** the backwards direction is trivial). So assume **Foundation**. Assume for a contradiction that there is x with $x \notin V$. Use **Separation** and **Union** and **Replacement** to form the set $TC(x)$ and $z = TC(x) \setminus V \supseteq x \setminus V$. If $z = \emptyset$ then $x \subseteq V$ and thus $x \in V$. Otherwise, let m be $\in$-minimal in z. Since $TC(x)$ is transitive and m is $\in$-minimal in z we must have $\forall t \in m\ t \in V$, i.e. $m \subseteq V$ giving $m \in V$ a contradiction.

Question 6

For **Union**: Suppose $x \in V$. Then $x \in V_\alpha$ for some least $\alpha \in \mathrm{On}$. Note that α must be a successor ordinal $\beta + 1$ (if α is a limit then $x \in \bigcup_{\beta \in \alpha} V_\beta$ so $x \in V_\beta$ for some $\beta \in \alpha$ contradicting minimality of α). Hence $x \subseteq V_\beta$. In U form $z = \bigcup^U x$. For $t \in z$ find $y \in x$ with $t \in y \in x \subseteq V_\beta$. Since V_β is transitive $t \in V_\beta$ and hence $z \subseteq V_\beta$. Thus $z \in V_\alpha$. Also, $z = \bigcup x$ is absolute so if $U \models z = \bigcup x$ and $x, z \in V$ then $V \models z = \bigcup x$.

For **Infinity**: We can either show (see next sheet) that $\omega \subseteq V_\omega$ so that $\omega \in V_{\omega+1}$. Or we show that V_ω is an inductive non-empty set: clearly $\emptyset \in V_1 \subseteq V_\omega$. If $x \in V_\omega$ then $x \in V_{n+1}$ for some $n \in \omega$, so $x \subseteq V_n \subseteq V_{n+1}$. Also $\{x\} \subseteq V_{n+1}$ so $x \cup \{x\} \subseteq V_{n+1}$ giving that $x \cup \{x\} \in V_{n+2} \subseteq V_\omega$. All of these operations are absolute, so V also believes that V_ω is inductive and non-empty. Finally $V_\omega \in V_{\omega+1} \subseteq V$.

Question 7

From lectures we have $\alpha \subseteq V_\alpha \cap \mathrm{On}$.

Now we inductively (on On) prove equality: this is clear for $\emptyset$. Suppose $V_\alpha \cap \mathrm{On} = \alpha$. If $\beta \in V_{\alpha+1} \cap \mathrm{On}$ then $\beta \subseteq \mathrm{On}$ and $\beta \subseteq V_\alpha$. Thus $\beta \subseteq V_\alpha \cap \mathrm{On} = \alpha$. Hence either $\beta \in \alpha$ or $\beta = \alpha$ so that in either case $\beta \in \alpha + 1$ as required.

For the second part, if $\alpha \in V_\beta$ then by (i) we must have $\alpha \in \beta$ (as the other cases lead to quick contradictions) so $\alpha + 1 \leq \beta$ giving $V_\alpha \in \mathcal{P}(V_\alpha) = V_{\alpha+1} \subseteq V_\beta$ since (from lectures) $\delta \leq \beta \to V_\delta \subseteq V_\beta$ (induction on β).

Question 8

See the separate document on General Recursion.