We first note that we can define the usual arithmetic functions on ω (interpreted as $\mathbb{N}$) by the Recursion Theorem and that these are absolute.

In the meta-theory, we then use a 'nice' Gödel numbering of the formulae of LST (although this is only relevant at the very end - but it does help understanding). Of course this does depend on our language, so we need to fix it: The terms are $v' \ldots '$ (or more formally we define recursively $t_0 = \{v'\}$, $t_{n+1} = \{s' : s \in t_n\}$) and we code them by

$$\lceil t \rceil = \begin{cases} 2; & t = v' \\ 2 \lceil s \rceil ; & t = s' \end{cases}$$

So 'terms' are powers of 2 and we write v_k instead of $v' \ldots '$ (k $'$s) (for sanity reasons) and we let $T = \left\{2^k : k \in \omega, k \geq 1\right\}$.

Next, the atomic formulae are (for t, s terms, so we can think $\lceil t \rceil , \lceil s \rceil \in T$)

$$t = s$$
$$t \in s$$

coded by

$$\lceil t = s \rceil = 3^{\lceil t \rceil} 5^{\lceil s \rceil} 7^1$$
$$\lceil t \in s \rceil = 3^{\lceil t \rceil} 5^{\lceil s \rceil} 7^2$$

and we let $A = \left\{3^t 5^s 7^k : t, s \in T, k \in \{1, 2\}\right\}$.

Finally, the formulae are

$$\begin{aligned} \phi; &\quad \phi \text{ an atomic formula} \\ \neg\phi; &\quad \phi \text{ a formula} \\ \phi \wedge \psi; &\quad \psi, \phi \text{ formulae} \\ \forall v_k \phi; &\quad v_k \text{ a term}, \phi \text{ a formula} \end{aligned}$$

coded by

$$\lceil \neg\phi \rceil = 3^{\lceil \phi \rceil} 7^3$$
$$\lceil \phi \wedge \psi \rceil = 3^{\lceil \phi \rceil} 5^{\lceil \psi \rceil} 7^4$$
$$\lceil \forall v_k \phi \rceil = 3^{\lceil \phi \rceil} 5^{\lceil v_k \rceil} 7^5$$

and we let

$$F = A \cup \left\{3^p 7^3 : p \in F\right\} \cup \left\{3^p 5^q 7^4 : p, q \in F\right\} \cup \left\{3^p 5^t 7^5 : p \in F, t \in T\right\}.$$

Of course, the definition of F doesn't seem to make sense, so we should (by recursion on ω) set

$$\begin{aligned} F_0 &= A \\ F_{n+1} &= F_0 \cup \left\{3^p 7^3 : p \in F_n\right\} \cup \left\{3^p 5^q 7^4 : p, q \in F_n\right\} \cup \left\{3^p 5^t 7^5 : p \in F_n, t \in T\right\} \\ F &= \bigcup_{n \in \omega} F_n. \end{aligned}$$

We note that T, A, F can be defined in a sufficiently large fragment of ZF − **Powerset** and are absolute for transitive non-empty transtive models of this fragment.

Now we define the function free on ω which takes values in $\omega^{<\omega}$ (the finite subsets of ω as follows (by recursion on ω):

$$\text{free}(0) = \{0\}$$

$$\text{free}(n+1) = \begin{cases} \{0\}\,; & n+1 \notin F \\ \{t,s\} & n+1 \in F \wedge n+1 = 3^k 5^s 7 \\ \{t,s\} & n+1 \in F \wedge n+1 = 3^k 5^s 7^2 \\ \text{free}(k); & n+1 \in F \wedge n+1 = 3^k 7^3 \\ \text{free}(k) \cup \text{free}(l); & n+1 \in F \wedge n+1 = 3^k 5^l 7^4 \\ \text{free}(k) \setminus \{l\}\,; & n+1 \in F \wedge n+1 = 3^k 5^l 7^5. \end{cases}$$

You should convince yourself that free gives $\{0\}$ if the input is not (the code for) a formula and otherwise the set of free variables in the formula.

(Note that I have made sure that $0 \notin T$ so that $0 \in \text{free}(k)$ if and only if $k \notin F$.)

We observe that free is absolute for non-empty transitive classes satisfying enough of ZF − **Powerset**.

Finally, given x, we can define a funcion $\text{val}_x \colon \omega \times x^{<\omega} \to \{0,1,2\}$ by recursion on ω (here I interpret $x^{<\omega} = \{a : b \to x : b \text{ finite } \subset \omega\}$).

$$\text{val}_x(0,a) = 2$$

$$\text{val}_x(n+1,a) = \begin{cases} 2; & n+1 \notin F \\ 2; & n+1 \in F \wedge \text{free}(n+1) \not\subseteq dom(a) \\ 0; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^1 \wedge a(k) \neq a(l)\right] \\ 1; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^1 \wedge a(k) = a(l)\right] \\ 0; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^2 \wedge a(k) \notin a(l)\right] \\ 1; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^2 \wedge a(k) \in a(l)\right] \\ 0; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 7^3 \wedge \text{val}_x(k,a) = 1\right] \\ 1; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 7^3 \wedge \text{val}_x(k,a) = 0\right] \\ 0; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^4 \wedge \right. \\ & \quad \left.[\text{val}_x(k,a) = 0 \vee \text{val}_x(l,a) = 0]\right] \\ 1; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^4 \wedge \right. \\ & \quad \left.[\text{val}_x(k,a) = 1 \wedge \text{val}_x(l,a) = 1]\right] \\ 0; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^5 \wedge \right. \\ & \quad \left. \exists \hat{a} \in x^{<\omega} \left[\hat{a}|_{\text{free}(n+1)\setminus\{l\}} = a|_{\text{free}(n+1)\setminus\{l\}} \wedge l \in dom(\hat{a}) \to \text{val}_x(k,\hat{a}) = 0\right]\right] \\ 1; & n+1 \in F \wedge \text{free}(n+1) \subseteq dom(a) \wedge \exists k,l \in \omega \left[n+1 = 3^k 5^l 7^5 \wedge \right. \\ & \quad \left. \forall \hat{a} \in x^{<\omega} \left[\hat{a}|_{\text{free}(n+1)\setminus\{l\}} = a|_{\text{free}(n+1)\setminus\{l\}} \wedge l \in dom(\hat{a}) \to \text{val}_x(k,\hat{a}) = 1\right]\right] \end{cases}$$

Note that because $x^{<\omega}$ is absolute (for transitive non-empty classes satisfying enough of ZF$-$**Powerset**), val_x is in fact absolute for these transitive non-empty classes.

For a formula $\phi(v_{k_1}, \ldots, v_{k_n})$ of LST with all free variables shown, and $a_1, \ldots, a_n \in x$ we define

$$(x, \in) \models \phi(a_1, \ldots, a_n) \equiv \mathrm{val}_x(\ulcorner\phi\urcorner, \{\langle k_i, a_i\rangle : i = 1, \ldots, n\}) = 1.$$

We now need to prove (by induction on the complexity of the formula) in the meta-theory that if A is a transitive, non-empty class satisfying enough of ZF $-$ **Powerset** then for every formula $\phi(v_{k_1}, \ldots, v_{k_n})$ of LST with all free variables shown

$$\mathrm{ZF} - \mathbf{Powerset} \vdash \forall a_1, \ldots, a_n \in x\, [\phi(a_1, \ldots, a_n)^x \leftrightarrow (x, \in) \models \phi(a_1, \ldots, a_n)]^A .$$

This is the 'standard' model theoretic proof that syntactic truth and semantic truth coincide.