Scientific Computing for DPhil Students II
Assignment 3

Due at lecture at 11:00 on Tuesday, 19 February 2019.

1. *Advection-diffusion equation in 1D.* Modify one of our MATLAB codes to solve

$$u_t = u_{xx} + 10u_x, \quad -1 < x < 1, \quad 0 < t < 0.125, \quad u(x,0) = e^{-10x^4/(1-x^2)}, \quad u(-1) = u(1) = 0.$$

Plot the solution at $t = 0.125$ and determine $u(-0.75, 0.125)$ to at least four digits of accuracy. (You can check your result with `pde15s` or `chebgui` — look at the scalar PDE demos.)

2. *Leap frog for the heat equation.* In 1910 L. F. Richardson proposed a discretization of $u_t = u_{xx}$:

$$\frac{v_j^{n+1} - v_j^{n-1}}{2k} = \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2}. \qquad (*)$$

The computers available in those days (viz., human beings) were slow, and Richardson did not detect that his formula was unstable. (a) Show by considering Fourier mode solutions $v_j^n = (g(\xi))^n \exp(i\xi x)$ that $(*)$ is unstable, and determine the largest unstable amplification factor $g = \max_\xi |g(\xi)|$. (b) Adapt one of our MATLAB codes to illustrate the instability computationally. In particular, take $h = 0.05$, $k = 0.001$, and $-1 < x < 1$ with boundary conditions $u(-1) = u(1) = 0$ and initial condition $u(x,0) = \exp(-10x^4/(1-x^2))$. Use $u(x,0)$ to define values v_j^0 and take one step of Crank-Nicolson to define values v_j^1. Now march forward with Richardson's scheme to steps v_j^{20} and v_j^{40}. Plot these functions, and report how well g and G agree with G defined by

$$G = (\max_j |v_j^{40}| / \max_j |v_j^{20}|)^{1/20}.$$

3. *The Gray-Scott equations.* Type this code into your computer:

```
function u00 = gs(N);
tic, S = spinop2('gs');
for i = 1:3
  dt = 2^(3-i);
  u = spin2(S,N,dt,'plot','off');
  subplot(1,3,i)
  plot(u{1}-.5,'zebra'), axis square off
  u00 = u{1}(0,0);
  s = sprintf('u(0,0) = %8.6f\n',u00);
  title(s,'fontsize',8), drawnow
end
toc
```

(a) Run `gs(32)`, `gs(64)`, and `gs(128)` and show the results.

(b) Describe what this code attempts to compute. Write down exactly what equation is being solved, including numerical values of the coefficients. Describe the meaning of the images and the meaning of the output value.

(c) Based on these and perhaps further experiments, discuss the observed convergence as a function of N and Δt. To three digits, what do you think is the correct value of `u00`, and approximately what choices of N and Δt are needed to achieve this?

(d) Repeat part *(a)* with a new code `gsspots` in which the string `'gs'` above is replaced by `'gsspots'`. What values of the coefficients are being used now?

(e) Repeat part *(a)* again with a code corresponding to coefficient values exactly three-quarters of the way from those of *(a)* to those of part *(d)*. Comment on the results.