Lecture 5, Sci. Comp. for DPhil Students II

Nick Trefethen, Tuesday 29.01.19

Last lecture

- IV.12 Stability regions
- IV.13 Stiffness
- IV.14 BVPs in Chebfun

Today

- V.1 PDEs in science and engineering
- V.2 Explicit 1D finite differences

Return Assignment 1

Assignment 2 due next Tuesday

Handouts

- Examples of PDEs
- Heat equation page from the *PDE Coffee Table Book*
- `m36_heat.m` & `m37_brownian.m` - heat equation and random walk
- `m38_rectangle10.m` - random walk in a 10x1 rectangle
- `m39_FisherKPP.m`, `m39chebfun.m` - Fisher-KPP equation
- Fisher-KPP equation page from the *PDE Coffee Table Book*

We've finished with ODEs and now turn to

V PDEs

PDE = partial differential equation: ≥ 2 independent variables

V.1 PDEs in science and engineering

Late 17c: calculus
19c: PDEs all over physics, especially linear
20c: the nonlinear explosion and spread into physiology, biology, electrical engineering, finance, and just about everywhere else

Pass around Folland and John books (these are at the pure maths end)

Notation: $u_t = \partial u/\partial t$, $u_{xx} = \partial^2 u/\partial x^2$, etc.

$\nabla u = (u_x, u_y, u_z)^T$ the **gradient** (a vector)

$\Delta u = u_{xx} + u_{yy} + u_{zz}$ the **Laplacian** (a scalar)

Hand out: sample pages from *PDE Coffee Table Book*
(a group project based at Oxford some years ago, never finished but available via "Books" at my home page at `https://people.maths.ox.ac.uk/trefethen/pdectb.html`)

"Examples of PDEs" sheet

Can you guess which three of these PDEs won Nobel Prizes?

Schrödinger, physics 1933
Hodgkin-Huxley, physiology or medicine 1963 (with Eccles)
Black-Scholes, economics 1997 (prize to Merton and Scholes; Black died in 1995)

Some examples from the sheet:

Laplace equation $\Delta u = 0$ (elliptic)

Poisson equation $\Delta u = f(x, y, z)$ (elliptic)

Heat or *diffusion equation* $u_t = \Delta u$ (parabolic)

Wave equation $u_{tt} = \Delta u$ (hyperbolic)

Burgers equation $u_t = (u^2)_x + u_{xx}$

KdV equation $u_t = (u^2)_x + u_{xxx}$

Constants have been omitted from all these equations — they are in nondimensional form.

V.2 Explicit 1D finite differences

The simplest approach to numerical solution of PDEs is finite difference discretization in both space and time (if it's time-dependent).

Let's consider the **heat equation** in 1D:

$$u_t = u_{xx}, \quad -1 < x < 1, \quad u(x,0) = u_0(x), \quad u(-1) = u(1) = 0.$$

Set up a regular grid with k = time step , h = space step

$$v_j^n \approx u(x,t)$$

Simplest finite difference formula:

$$\frac{v_j^{n+1} - v_j^n}{k} = \frac{v_{j+1} - 2v_j^n + v_{j-1}^n}{h^2}$$

```
          X
          |
    X-----X-----X

       "stencil"
```

Consider v^n as an N-vector. We can get from v^n to v^{n+1} by multiplication by a tridiagonal matrix A:

```
              | a    b                  |  |  n |
              |                         |  | v  |
              | b    a    b             |  |  1 |
              |                         |  |    |
 n+1      n   |    .    .   .           |  |    |
v     = Av  = |     .    .   .          |  |    |
              |      .    .   .         |  |    |
              |                         |  |    |
              |         b    a    b     |  |  n |
              |                         |  | v  |
              |              b    a     |  |  N |
```

where $a = 1 - 2(k/h^2)$, $b = k/h^2$. Note that we have implicitly imposed the conditions $v_0^n = v_{N+1}^n = 0$.

You can program this as a for/do loop, or via sparse matrices.

[`m36_heat.m` - finite differences for the heat equation]

Discuss in this code:

- how the boundary conditions are imposed

Explore:

- instability if k is increased (`m36u`)
- step-by-step vs. movie (`m36pause`)
- large-t changes if periodic BCs (`m36per`)

[`m37_brownian.m` - random walk / Brownian motion — the physics underlying the heat equation]

[also `m38_rectangle.m`, illustrating Brownian motion in a 10×1 rectangle]

How we move around between discrete and continuous in physics and numerics!

```
DISCRETE                    CONTINUOUS

bouncing molecules          continuum models of physics
finite-difference approx    PDE
random walk                 Brownian motion
dots on computer screen     our perception of movie
floating-point arithmetic   real arithmetic
```

Somehow the first column is generally the "truth" in a literal sense, while the second is the "truth" in a deeper (conceptual) sense

Note that these are all happening on different scales, which are not in general tied to one another.

The heat equation is linear, dating to Fourier in 1807, and it can be more or less solved analytically.

Let's consider now a nonlinear PDE, intractable analytically but easy to solve numerically:

Fisher-KPP equation

Independent 1937 discoveries for biological applications (spread of species): Fisher; Kolmogorov, Petrovsky, and Piscounov

Handout: page from the *PDE Coffee Table Book.*

Solutions: traveling waves. Similar more complicated behaviour: Hodgkin-Huxley eqs for neural impulses (1952).

$$u_t = \epsilon u_{xx} + u - u^2$$

Explicit finite difference model:

$$\frac{v_j^{n+1} - v_j^n}{k} = \epsilon \frac{v_{j+1} - 2v_j^n + v_{j-1}^n}{h^2} + (v_j - v_j^2)$$

For our MATLAB code we have BCs $u(0) = 1$, $u(10) = 0$. The former is done by use of an extra vector “bc”:

```
            | a    b                   |   |  n |      | b |
            |                          |   | v  |      |   |
            | b    a    b              |   |  1 |      | 0 |
            |                          |   |    |      |   |
 n+1        |     .    .    .          |   |    |      |   |                 2
v      =    |      .    .    .         |   |    |  +   |   |   +  k (v -v )
            |       .    .    .        |   |    |      | : |            j  j
            |                          |   |    |      |   |
            |          b    a    b     |   |  n |      |   |
            |                          |   | v  |      |   |
            |               b    a     |   |  N |      | 0 |
```

[`m39_FisherKPP.m`, `m39u.m`, `m39chebfun.m`]