Lecture 10, Sci. Comp. for DPhil Students II

Nick Trefethen, Tuesday 19.02.19

Last lecture

- V.9 Fourier spectral discretisation
- V.10 Fourier spectral discretisation via FFT

Today

- V.11 Fourier, Laurent, and Chebyshev
- V.12 Chebyshev series and interpolants

Handouts

- Assignment 3 solutions
- Assignment 4
- p 1 of Salzer, barycentric interpolation, 1972
- p 1 of Berrut & T, same topic, 2004
- `m53_series.m`
- `m54_bernstein.m`

Assignment 3 due today

Pass around a copy of *Approximation Theory and Approximation Practice* (“*ATAP*”)

V.11 Fourier, Laurent, and Chebyshev

These are three parallel worlds that are the basis of all kinds of practical mathematics, including spectral methods for ODEs and PDEs.

Fourier is identical to Laurent via $z = e^{it}$.

Chebyshev is almost identical to these via $x = (z + z^{-1})/2 = \cos(t)$, but not quite identical, because it entails an assumption of $t \leftrightarrow -t$ or $z \leftrightarrow z^{-1}$ symmetry.

FOURIER

Periodic function $F(t)$, $t \in [0, 2\pi]$

$2n+1$ equispaced points: $t_k = 2\pi k/(2n+1)$, $0 \le k \le 2n$

Complex exponential e^{ikt}

Trigonometric interpolant $P_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$

Quadrature: trapezoidal rule $\Leftrightarrow$ integrating the interpolant

Rootfinding: via eigenvalues of companion matrix

Limit $n \to \infty$: Fourier series $F(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt}$

Fourier coefficient: $a_k = (1/2\pi) \int_0^{2\pi} f(t) e^{-ikt} dt$

F analytic in a strip $\Rightarrow a_k = O(C^{-|k|})$

Reference: Wright, Javed, Montanelli & T., "Extension of Chebfun to periodic functions", *SINUM* 2016, at my web page

LAURENT

Function $\mathcal{F}(z)$, $z \in$ unit circle

$2n+1$ roots of unity: $z_k = \exp(it_k)$, $0 \le k \le 2n$

Monomial z^k

Laurent polynomial interpolant $\mathcal{P}_n(z) = \sum_{k=-n}^{n} c_k z^k$

Quadrature: trapezoidal rule $\Leftrightarrow$ integrating the interpolant

Rootfinding: via eigenvalues of companion matrix

Limit $n \to \infty$: Laurent series $\sum_{k=-\infty}^{\infty} a_k z^k$

Laurent coefficient: $a_k = (1/2\pi i) \int \mathcal{F}(z) z^{-k-1} dz$ over unit circle

$\mathcal{F}$ analytic in an annulus $\Rightarrow a_k = O(C^{-|k|})$

Reference: Austin, Kravanja & T., "Numerical algorithms based on analytic function values at roots of unity", *SIAM J. Numer. Anal.*, 2014

CHEBYSHEV

Function $f(x)$, $x \in [-1,1]$

$n+1$ Chebyshev points: $x_k = \cos(k\pi/n)$, $0 \le k \le n$

Chebyshev polynomial $T_k(x) = (z^k + z^{-k})/2 = \cos(k\cos^{-1} x)$

Polynomial interpolant $p_n(x) = \sum_{k=0}^{n} c_k T_k(x)$

Quadrature: Clenshaw-Curtis formula $\Leftrightarrow$ integrating the interpolant

Rootfinding: via eigenvalues of colleague matrix

Limit $n \to \infty$: Chebyshev series $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$

Chebyshev coefficient: $a_k = (2/\pi) \int_{-1}^{1} f(x) T_k(x) dx / \sqrt{1-x^2}$

f analytic in an ellipse $\Rightarrow a_k = O(C^{-k})$

Reference: T, *Approximation Theory and Approximation Practice,* SIAM, 2013

[`m53_series.m`]

V.12 Chebyshev series and interpolants

This material is so fundamental, and so often unfamiliar even to those who need it, that we're going to be a little more academic than usual and state five theorems.

The reference for all of this material is *Approximation Theory and Approximation Practice.* (Pass it around.)

Chebyshev polynomials

First, a reminder about Chebyshev polynomials. If $x = \cos t$, we have $T_k(x) = \cos(kt) = \cos(k\cos^{-1}(x))$. In particular, one finds

$$T_0(x) = \cos(0t) = 1, \quad T_1(x) = \cos(1t) = x,$$

$$T_2(x) = \cos(2t) = 2x^2 - 1 \quad T_3(x) = \cos(3t) = 4x^2 - 3x$$

and in general

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).$$

Chebyshev series

Theorem 1. *Let f be Lipschitz continuous on $[-1,1]$. Then if the Chebyshev coefficients of f are defined by*

$$a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}dx,$$

except with $1/\pi$ instead of $2/\pi$ for a_0, then the series

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

converges absolutely and uniformly.

Absolute and uniform convergence imply that you can reorder the terms however you like and get the same result; also the series still converges if you take absolute values.

Bernstein ellipse

How fast does the Chebyshev series converge? This depends on how smooth f is. Let's suppose f is *analytic* on $[-1,1]$, i.e., has a convergent Taylor series at each point of $[-1,1]$. Then it can be analytically continued some distance into the complex x-plane. Specifically, it can be extended to a function satisfying

$$|f(x)| \leq M, \quad x \in E_\rho$$

for some constant M in the **Bernstein ρ- ellipse** for some $\rho > 1$, defined as the ellipse in the x-plane with foci ± 1 whose semimajor and semiminor axis lengths sum to ρ. Equivalently, it is the ellipse that is the image of the circle $|z| = \rho$ under the Joukowski transformation $x = (z + z^{-1})/2$.

(sketch)

Decay rate of coefficients

This theorem is due to Bernstein in 1912. The standard proof makes use of contour integrals.

Theorem 2. *Let f be analytic with $|f(x)| \leq M$ in the Bernstein ρ -ellipse for some $\rho > 1$. Then for each $k \geq 0$,*

$$|a_k| \leq 2M\rho^{-k}$$

Convergence rate of Chebyshev series

This theorem is a corollary of Theorem 2, also due to Bernstein in 1912.

Theorem 3. *Let $f_n(x)$ be the truncation of the Chebyshev series for f at term n. Then under the assumption of Theorem 2, for each $n \geq 0$,*

$$\|f - f_n\|_\infty \leq \frac{2M\rho^{-n}}{\rho - 1}$$

Chebyshev interpolants

Given f, there is a unique polynomial interpolant p_n of degree at most n through f at the $n+1$ Chebyshev points. We call this the degree n **Chebyshev interpolant** of f.

Convergence rate of Chebyshev interpolants

The proof of the next theorem is pretty straightforward, based on Theorem 2 combined with principles of *aliasing*, but we won't go into this.

Theorem 4. *Under the assumption of Theorem 2, for each $n \geq 0$,*

$$\|f - p_n\|_\infty \leq \frac{4M\rho^{-n}}{\rho - 1}$$

Computation by the barycentric formula

To compute a Chebyshev interpolant, one can find its Chebyshev coefficients via the FFT and then just evaluate the series.

But there is also a beautiful, numerically stable formula for doing this "by values" rather than "by coefficients". It is called the *barycentric formula*, and it is due to Salzer in 1972. (The ± 1 coefficients in the formula were derived earlier by Marcel Riesz in 1916.)

Throughout the 20th century there have been widespread misconceptions about polynomial interpolation. Many books tell you it can't be done very reliably numerically, or that you need to use the Newton form of the interpolant. This is incorrect. In fact the Lagrange form is better for most purposes, and the barycentric formula is of Lagrange form.

Theorem 5. *The following formula gives the Chebyshev interpolant p_n to f.*

$$p_n(x) = \sum_{j=0}^{N}{}' \frac{(-1)^j f(x_j)}{x - x_j} \Bigg/ \sum_{j=0}^{N}{}' \frac{(-1)^j}{x - x_j}$$

The prime means that terms $j = 0$ and $j = N$ are multiplied by $1/2$. If $x = x_j$, we set $p_n(x) = f(x_j)$.

See Salzer 1972 and Berrut and Trefethen 2004 handouts.

For $x = x_j$, the formula has a $0/0$ division. Surely it must be numerically unstable in floating-point arithmetic for $x \approx x_j$, because of cancellation error? No! It is perfectly stable, essentially because cancellation errors in the numerator match cancellation errors in the denominator). This has been proved by N. J. Higham, *IMA J. Numer. Anal.*, 2004.

[`m54_bernstein.m`]