Interactions

In order to incorporate interactions we need to add a potential term to the Lagrangian:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \sum_{n=3} \frac{1}{n!} \lambda_n \phi^n$$

λ_n ← coupling constants

It is a local Lagrangian — depends on fields and their derivatives at one point (avoid faster than light travel)

- Various kinds of interactions:

$$[m] = 1, \quad [x] = -1, \quad [dx] = -1, \quad [d^4x] = -4$$

$$[S] = 0 \Rightarrow [\mathcal{L}] = 4$$

Then (in four dimensions):

$$[\lambda_n] = 4 - n$$

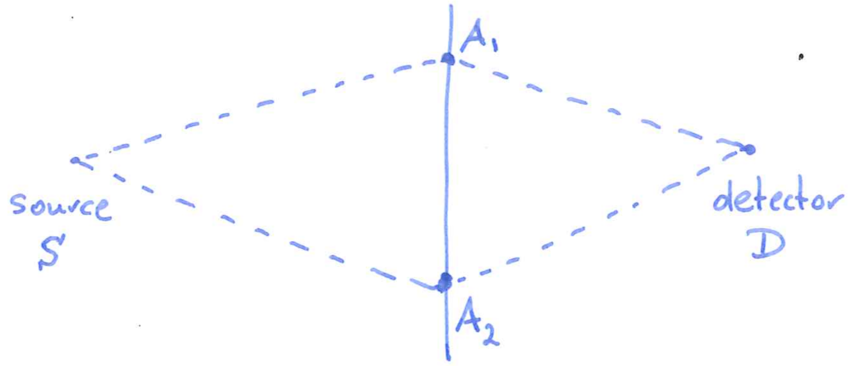
We distinguish three cases:

- $[\lambda_3] = 1 > 0$: important at small energies but not at high energies (relevant perturbation)
- $[\lambda_4] = 0$: (marginal perturbation)
- $[\lambda_n] < 0$ for $n > 4$: important at high energies → leads to non-renormalisibility (irrelevant perturbation)

Path integral formulation of QM

- Motivation : double-slit experiment

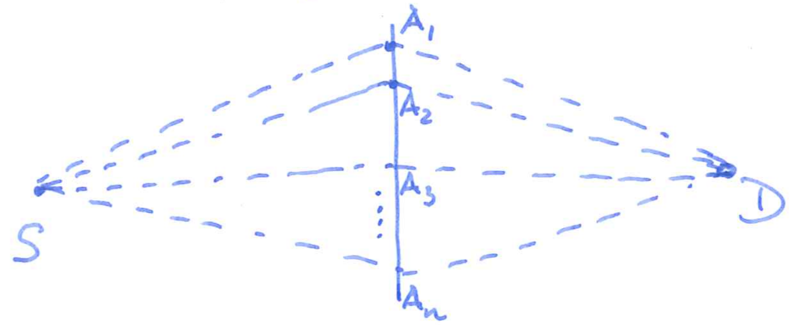
① two holes, one screen



$$\mathcal{A}_{\text{detected}} = \mathcal{A}(S \to A_1 \to D) + \mathcal{A}(S \to A_2 \to D)$$

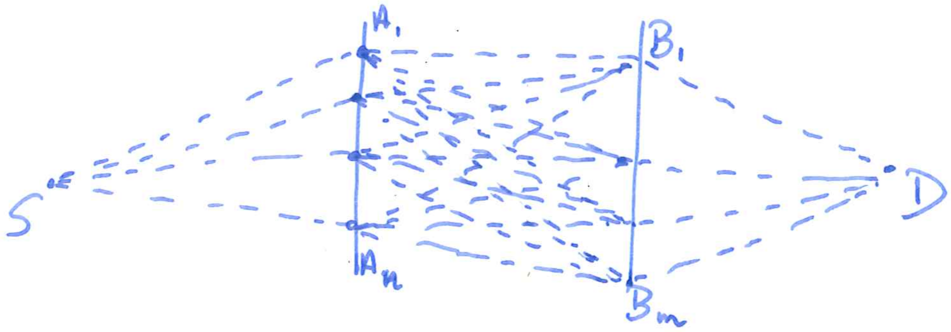
superposition principle of QM

② many holes, one screen



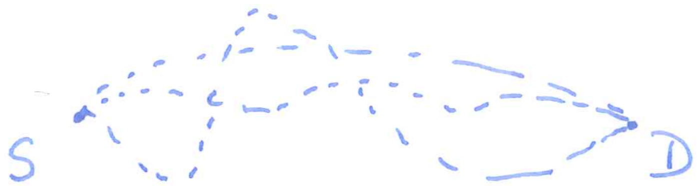
$$\mathcal{A}_{\text{detected}} = \sum_{i=1}^{n} \mathcal{A}(S \to A_i \to D)$$

③ many holes, many screens



$$\mathcal{A}_{\text{detected}} = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathcal{A}(S \to A_i \to B_j \to D)$$

④ Infinitely many screens and holes



$$\mathcal{A}_{\text{detected}} = \sum_{\text{paths}} \mathcal{A}(\text{particle goes from } S \text{ to } D \text{ in time } T \text{ along path})$$

- How to construct such amplitudes over paths?

  Amplitude to propagate from state q_I to state q_F in time T

  $$M = \langle q_F | e^{-i\hat{H}T} | q_I \rangle \quad , \; T = t_F - t_I$$

  where $\hat{H}$ is the Hamiltonian

- More precisely (one-dimensional case)
  - split the interval (t_I, t_F) into smaller pieces

    $$(t_I, t_1, t_2, \dots, t_{n-1}, t_F) \quad \text{with} \quad t_{j+1} - t_j = \underset{\uparrow \text{fixed}}{\Delta t}$$

  $$M = \langle q_F | e^{-i\hat{H}\Delta t} \dots e^{-i\hat{H}\Delta t} | q_I \rangle$$

  - insert a complete set of eigenstates

    $$1 = \int_{-\infty}^{+\infty} dq(t_j) \, |q(t_j)\rangle\langle q(t_j)| \qquad \forall j$$

  $$M = \prod_j \int dq(t_j) \langle q(t_{j+1}) | e^{-i\hat{H}\Delta t} | q(t_j) \rangle$$

  - consider just the kinetic term first : $\hat{H} = \frac{\hat{p}^2}{2m}$

    and introduce a complete set of eigenvalues

    $$1 = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \, |p\rangle\langle p|$$

Then

$$\langle q(t_{j+1}) | e^{-i\hat{H}\Delta t} | q(t_j)\rangle$$

$$= \int \frac{dp}{2\pi} \langle q(t_{j+1}) | e^{-i\frac{\hat{p}^2}{2m}\Delta t} | p\rangle \langle p | q(t_j)\rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\Delta t} \langle q(t_{j+1}) | p\rangle \langle p | q(t_j)\rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\Delta t} e^{ip(q(t_{j+1}) - q(t_j))}$$

$$\underset{\text{integrals}}{\overset{\text{Fresnel}}{=}} \left(\frac{-im}{2\pi\Delta t}\right)^{1/2} e^{i\Delta t \frac{m}{2}\left(\frac{q(t_{j+1}) - q(t_j)}{\Delta t}\right)^2}$$

Then

$$M = \left(\frac{-im}{2\pi\Delta t}\right)^{n/2} \prod_{i=1}^{n-1} \int dq(t_i)\, e^{i\Delta t \frac{m}{2}\sum_{j=0}^{n-1}\left(\frac{q(t_{j+1}) - q(t_j)}{\Delta t}\right)^2}$$

- Continuum limit $\Delta t \to 0 \Rightarrow \begin{cases} \frac{q(t_{j+1}) - q(t_j)}{\Delta t} \longrightarrow \dot{q} \\ \Delta t \sum_{j=0}^{n-1} \longrightarrow \int_{t_I}^{t_F} dt \end{cases}$

$$M = \int \mathcal{D}q(t)\, e^{i\int_{t_I}^{t_F} dt\, \frac{m\dot{q}^2}{2}}$$

where $\int \mathcal{D}q(t) = \lim_{n\to\infty} \left(\frac{-im}{2\pi\Delta t}\right)^{n/2} \left(\prod_{k=1}^{n-1} \int dq_k\right)$

↑ integral over paths

- Reintroduce the potential

$$\langle q_F | e^{-i\hat{H}t} | q_I\rangle = \int \mathcal{D}q(t)\, e^{i\int_{t_I}^{t_F} dt\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)}$$

$$= \int \mathcal{D}q(t)\, e^{i\int_{t_I}^{t_F} dt\, L(q,\dot{q})}$$

$$= \int_{\substack{q(t_I)=q_I \\ q(t_F)=q_F}} \mathcal{D}q(t)\, e^{iS[q]}$$

- Classical limit: $\hbar \to 0$

$$\langle q_F | e^{-\frac{i}{\hbar}\hat{H}t} | q_I \rangle = \int \mathcal{D}q(t)\, e^{\frac{i}{\hbar}\int_{t_I}^{t_F} dt\, L(q,\dot{q})}$$

From the saddle-point method (method of steepest descent):

$$\langle q_F | e^{-\frac{i}{\hbar}\hat{H}t} | q_I \rangle_{\hbar \to 0} \simeq e^{\frac{i}{\hbar}\int_{t_I}^{t_F} dt\, L(q_c,\dot{q}_c)}$$

where q_c is the classical path obtained by solving the Euler-Lagrange eq.:

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{q}} - \frac{\delta \mathcal{L}}{\delta q} = 0$$

with appropriate boundary conditions.

- Semi-classical limit
  - important paths close to the classical one
  - for periodic classical paths: they carry a complex phase which averages to 0 after many periods
  - in case of orbits with $S = 2\pi\hbar \cdot$ integer this phase is unity → they dominate the path integral

Bohr - Sommerfeld quantization

- Imaginary time path integral:
  - due to i in the exponent, the path integral is oscillatory → problems with convergence
  - go to Euclidean time performing Wick rotation

$$\int \mathcal{D}q(t)\, e^{i\int dt\, L} \xrightarrow[t=-i\tau]{} \int \mathcal{D}q(\tau)\, e^{-\int d\tau\, L_E}$$

Comments on the imaginary path integral

Sometimes it is useful to consider matrix elements

$$M = \langle q_F | e^{-\hat{H}(\tau_F - \tau_I)} | q_I \rangle \qquad (*)$$

This leads to the path integral:

$$\int \mathcal{D}q \; e^{-S_E[q]}$$

with

$$S_E[q] = \int_{\tau_I}^{\tau_F} \left(\frac{1}{2}\dot{q}^2 + V(q(\tau)) \right) d\tau$$

This is called the imaginary-time path integral and can be formally obtained from the previous one by letting $t = -i\tau$

- Application: quantum statistical mechanics

  Canonical partition function:

$$Z = \operatorname{Tr} e^{-\beta \hat{H}} \quad \text{with } \beta = \frac{1}{k_B T}$$

The trace can be written as:

$$Z = \int dq_i \langle q_i | e^{-\beta \hat{H}} | q_i \rangle$$

It is of the form $(*)$ with $\tau_F - \tau_I = \beta$. Then Z can be written as the imaginary-time path integral over periodic paths satisfying $q(\tau_i + \beta) = q(\tau_i)$

- Quantum mechanics (in imaginary time)

  $\equiv$ classical statistical mechanics in one higher spatial dimension

Path integrals in QFT

- Lattice appoximation

Lattice in d dimensions

$(\hat{p},\hat{q})$ - one $\hat{p}$ and one $\hat{q}$ operators for each lattice site

$$\hat{q} \to (\hat{\phi}(x_1),\dots,\hat{\phi}(x_N))$$
$$\hat{p} \to (\hat{\Pi}(x_1),\dots,\hat{\Pi}(x_N))$$
with $[\hat{\phi}(x_j),\hat{\Pi}(x_{j'})] = i\delta_{jj'}$

- The simplest Hamiltonian

$$H = \sum_j \left(\frac{1}{2}\Pi(x_j)^2 + V(\phi(x_j))\right) + \frac{1}{2} J \sum_{(jj')} (\phi(x_j) - \phi(x_{j'}))^2$$

sum runs over nearest neighbours

- Path integral :

$$\int_{\substack{\phi(x_j,t_I)=\phi_I(x_j)\\ \phi(x_j,t_F)=\phi_F(x_j)}} \prod_j \mathcal{D}\phi(x_j,t)\, e^{iS[\phi(x_j,t)]}$$

with

$$S = \int dt \left[\sum_j \left(\frac{1}{2}\dot{\phi}(x_j,t)^2 - V(\phi(x_j,t))\right) - \frac{J}{2}\sum_{(jj')}(\phi(x_j,t)-\phi(x_{j'},t))^2\right].$$

This is the action of Lattice Field Theory.

- Continuum limit ($a \to 0$)

$$\sum_j \to \int \frac{d^d x}{a^d} \quad , \quad \sum_{(j,j')} (\phi(x_j,t)-\phi(x_{j'},t))^2 \to \int \frac{d^d x}{a^d} a^2 (\nabla\phi(x,t))^2$$

Rescale : $\phi \to J^{-1/2} a^{(d-2)/2} \phi$.

Then :

$$S = \int dt \int d^d x \left[\frac{1}{a^d}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) - \frac{J}{a^{d-2}}(\nabla\phi)^2 \right]$$

$$\to \int dt \int d^d x \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right)$$

Action for classical field theory

- Finally, we arrive to the quantum field theory path integral :

$$\boxed{\int \mathcal{D}\phi(\vec{x},t)\, e^{iS[\phi]}}$$

- Similar to the quantum mechanics there is a relation between path integral and propagation amplitudes:

$$\langle \phi_F | e^{-iHT} | \phi_I \rangle = \int_{\substack{\phi(\vec{x},t_I)=\phi_I(\vec{x}) \\ \phi(\vec{x},t_F)=\phi_F(\vec{x})}} \mathcal{D}\phi \, e^{iS[\phi]}$$

where the states $|\phi_I\rangle$ and $|\phi_F\rangle$ are defined as

$$\hat{\phi}(\vec{x})\,|\phi_F\rangle = \phi_F(\vec{x})\,|\phi_F\rangle$$

and

$$\hat{\phi}(\vec{x})\,|\phi_I\rangle = \phi_I(\vec{x})\,|\phi_I\rangle$$