Generating functionals

- On relativistic grounds, we should consider the limit of infinite time since we usually discuss the limit of infinite space → the only meaningful path integral is:

$$\int \mathcal{D}\phi \, e^{i\int_{-\infty}^{+\infty} dt \int d^d x \, \mathcal{L}}$$

which is just a number.

- In the Euclidean field theory we have

$$\int \mathcal{D}\phi \, e^{-\int_{-\infty}^{+\infty} d\tau \int d^d x \, \mathcal{L}_E} \qquad (*)$$

We expand the initial and final states in the basis of Hamiltonian eigenstates:

$$|\phi_I\rangle = \sum_n \alpha_n |n\rangle \quad , \quad |\phi_F\rangle = \sum_n \beta_n |n\rangle$$

and get

$$(*) \hat{\simeq} \lim_{t_F - t_I \to \infty} \sum_n \alpha_n \beta_n \, e^{-E_n(\tau_F - \tau_I)} \langle n | n \rangle$$

$$\sim e^{-E_0(\tau_F - \tau_I)} \langle \Omega | \Omega \rangle$$

where $|\Omega\rangle$ is the lowest energy state (vacuum) with energy E_0.

We have just got the vacuum-to-vacuum amplitude → not very interesting quantity

- In order to get interesting physical quantities we need to „tickle" the vacuum $\rightarrow$ add sources

$$S \longrightarrow S + \int J(x)\, \phi(x)\, d^{d+1}x$$

The vacuum amplitude is now a functional depending on this source function $J(x)$:

$$Z[J] = \int \mathcal{D}\phi\, e^{iS + i\int J(x)\phi(x)\, d^{d+1}x}$$

The „i" makes this integral ill-defined $\rightarrow$ we develop our theory in the Euclidean space instead:

$$Z[J] = \int \mathcal{D}\phi\, e^{-S_E + \int J(x)\phi(x)\, d^{d+1}x}$$

- Interesting physical quantities are found by taking functional derivatives of $Z[J]$ with respect to J.

<u>Example</u>:

$$-\frac{1}{Z[0]} \frac{\delta Z[J]}{\delta J(x_1)}\bigg|_{J=0} = \frac{1}{Z[0]} \int \mathcal{D}\phi\, \phi(x_1)\, e^{-S[\phi]}$$

$$\equiv \langle \phi(x_1) \rangle$$

We call it a correlation function in analogy with the statistical mechanics.

$$-\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x_1)\, \delta J(x_2)}\bigg|_{J=0} = \frac{1}{Z[0]} \int \mathcal{D}\phi\, \phi(x_1)\, \phi(x_2)\, e^{-S[\phi]}$$

$$\equiv \langle \phi(x_1)\, \phi(x_2) \rangle$$

- What is the meaning of these quantities in the operator description? Take $\tau_I \to -\infty$, $\tau_F \to +\infty$. Then

$$\int \mathcal{D}\phi \; \phi(x_1) e^{-S[\phi]} \sim e^{-E_0(\tau_F-\tau_1)} e^{-E_0(\tau_1-\tau_I)} \langle \Omega | \hat{\phi}(\vec{x}_1) | \Omega \rangle$$

and

$$\langle \phi(x_1) \rangle \sim \frac{e^{-E_0(\tau_F-\tau_I)} \langle \Omega | \hat{\phi}(\vec{x}_1) | \Omega \rangle}{e^{-E_0(\tau_F-\tau_I)} \langle \Omega | \Omega \rangle}$$

$$= \langle \Omega | \hat{\phi}(\vec{x}_1) | \Omega \rangle$$

Similar for two-point correlation functions

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{1}{Z[0]} \int \mathcal{D}\phi \; \phi(x_1) \phi(x_2) e^{-S[\phi]}$$

$$\left\{ \int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\substack{\phi(x_1^0,\vec{x}) = \phi_1(\vec{x}) \\ \phi(x_2^0,\vec{x}) = \phi_2(\vec{x})}} \mathcal{D}\phi(x) \right\}$$

$$\overset{x_1^0 < x_2^0}{=} \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \langle \phi_F | e^{-\hat{H}(\tau_F - x_2^0)} \hat{\phi}(\vec{x}_2) | \phi_2 \rangle \cdot$$

$$\cdot \langle \phi_2 | e^{-\hat{H}(x_2^0 - x_1^0)} \hat{\phi}(\vec{x}_1) | \phi_1 \rangle \langle \phi_1 | e^{-\hat{H}(x_1^0 - \tau_I)} | \phi_I \rangle$$

$$= \langle \phi_F | e^{-\hat{H}(\tau_F - x_2^0)} \hat{\phi}(\vec{x}_2) e^{-\hat{H}(x_2^0 - x_1^0)} \hat{\phi}(\vec{x}_1) e^{-\hat{H}(x_1^0 - \tau_I)} | \phi_I \rangle$$

$$\xrightarrow[\text{picture}]{\text{Heisenberg}} \langle \phi_F | e^{-\hat{H}\tau_F} \hat{\phi}(x_2) \hat{\phi}(x_1) e^{\hat{H}\tau_I} | \phi_I \rangle$$

$$\xrightarrow[\tau_I \to -\infty]{\tau_F \to +\infty} e^{-E_0(\tau_F - \tau_I)} \langle \Omega | \hat{\phi}(x_2) \hat{\phi}(x_1) | \Omega \rangle$$

Similar for $x_1^0 > x_2^0$. Finally:

$$\langle \phi(x_1) \phi(x_2) \rangle = \langle \Omega | T\{ \hat{\phi}(x_1) \hat{\phi}(x_2) \} | \Omega \rangle$$

- Functional derivatives of $Z[J]$ give vacuum expectation values of time-ordered products of field operators

- In field theory, the correlation functions are called Green's functions or simply N-point functions

$$G^{(N)}(x_1, \dots, x_N) = \langle \phi(x_1) \dots \phi(x_N) \rangle = \frac{1}{Z[0]} \frac{\delta^N Z[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0}$$

Equivalently

$$\frac{Z[J]}{Z[0]} = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^{d+1}x_1 \dots d^{d+1}x_N \, G^{(N)}(x_1 \dots x_N) J(x_1) \dots J(x_N)$$

$Z[J]$ is called the generating function for the N-point functions.

- It is also useful to define

$$W[J] \equiv \log Z[J]$$

Analogous to the free energy in statistical mechanics.

We define:

$$G_c^{(N)}(x_1, \dots, x_N) = \frac{\delta^N W[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0}$$

These are so-called connected correlation functions.

Example:

$$\frac{\delta^2 W[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} = \langle \phi(x_1) \phi(x_2) \rangle_c = \langle \phi(x_1) \phi(x_2) \rangle - \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle$$

Propagator in free theory

- Free scalar in Euclidean space: $\mathcal{L}_E = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2$

$$Z_0[J] = \int \mathcal{D}\phi \, e^{-\int [\underbrace{\frac{1}{2}\partial^\mu\phi\partial_\mu\phi}_{= -\frac{1}{2}\phi\partial^\mu\partial_\mu\phi} + \frac{1}{2}m^2\phi^2] d^{d+1}x + \int J(x)\phi(x)\, d^{d+1}x}$$

- In Fourier space

$$\tilde{\phi}(p) = \int d^{d+1}x \, e^{-ip\cdot x}\phi(x), \quad \phi(x) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} e^{ipx}\tilde{\phi}(p)$$

- The exponent of $Z_0[J]$ in Fourier space

$$-\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left[\frac{1}{2}\tilde{\phi}(p)(p^2+m^2)\tilde{\phi}(-p) - \tilde{J}(p)\tilde{\phi}(-p)\right]$$

$$= -\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \Big[\frac{1}{2}\left(\tilde{\phi}(p) - \frac{1}{p^2+m^2}\tilde{J}(p)\right)(p^2+m^2)\left(\tilde{\phi}(-p) - \frac{1}{p^2+m^2}\tilde{J}(-p)\right)$$
$$- \frac{1}{2}\tilde{J}(p)\frac{1}{p^2+m^2}\tilde{J}(-p)\Big]$$

- We change variables: $\tilde{\phi}(p) \longrightarrow \tilde{\phi}(p) + \frac{1}{p^2+m^2}\tilde{J}(p)$

$$Z_0[J] = \int \mathcal{D}\tilde{\phi}\, e^{-\frac{1}{2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}}\tilde{\phi}(p)(p^2+m^2)\tilde{\phi}(-p) + \frac{1}{2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}}\tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

$$= Z_0[0]\, e^{\frac{1}{2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}}\tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

- Go back to the coordinate space

$$Z_0[J] = Z_0[0]\, e^{\frac{1}{2}\int d^{d+1}x'\, d^{d+1}x''\, J(x')\Delta(x'-x'')J(x'')}$$

with $$\Delta(x'-x'') = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{ip(x'-x'')}}{p^2+m^2}$$